SUPERPROCESSES

By

HRVOJE ŠIKIĆ

A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL
OF THE UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

1993

ACKNOWLEDGEMENTS

Grateful acknowledgements are due to persons whose help and support made this work possible. First, and foremost, my advisor Prof. Murali Rao is thanked for his guidance throughout my study. His original insight into mathematics and his humanity are a constant inspiration for me. Prof. Zoran R. Pop-Stojanović already directed some of my work in Zagreb. His vast knowledge made mathematical discussions particularly enjoyable. Prof. Joseph Glover helped me very efficiently on numerous occasions, ranging from research questions to organizational problems. I also wish to thank Prof. James K. Brooks and Prof. Andrew J. Rosalsky for their service on my committee.

I wish to thank faculty members of the Department of Mathematics of the University of Zagreb for giving me a strong mathematical foundation. In particular, I am grateful to Prof. Nikola Sarapa, who introduced me to probability and led me to magister degree, and to my longtime friend and colleague Dr. Zoran Vondraček, with whom I worked from our first mathematical steps until today.

And last, but not least, I wish to express my gratitude to the members of my family. My parents, whose working attitude and understanding of criticism as the basic approach to science, are lasting examples for me. My wife Sonja, who, being a mathematician herself, was a supportive and understanding companion in the sometimes very demanding life of a mathematician.

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Abstract of Dissertation Presented to the Graduate School
of the University of Florida in Partial Fulfillment of the
Requirements for the Degree of Doctor of Philosophy

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HRVOJE ŠIKIĆ

May 1993

Chairman: Dr. Murali Rao
Major Department: Mathematics

Dawson-Watanabe superprocesses (X_t), over a Markov process (ξ_t), are usually defined by transition supersemigroups $Q_t(\mu, d\nu)$. The Laplace transform of Q_t is determined by a nonlinear semigroup (V_t), which is obtained as the perturbation of Markov semigroup (T_t) (given by ξ) by a branching mechanism Ψ.

In the first part of this work we establish a general analytical framework for nonlinear perturbations of positive semigroups, which preserve positivity. The semigroups involved are only weakly measurable with respect to the subspace M of the space of linear functionals. The well-known results on superprocesses by Fitzsimmons and Watanabe are obtained as special cases. In this fashion (X_t) can be defined as M-valued process, in general. Very useful and new, Fubini-type lemma is applied several times in this part; among other things to estimate (V_t) from below and above in terms of the Kac semigroup of (T_t).

In the second part we analyze (X_t) through (V_t) and (Q_t). We start by detailed account of one-dimensional superprocesses (studied by Lamperti in the late sixties). Some new formulas for (V_t) and (Q_t) are derived. These results are then used to give

detailed description of "component processes" (X_t^{Ψ}) (which depends only on Ψ) and (X_t^{ξ}) (which depends only on ξ). The new notion of a measure on rays is introduced to obtain the description of (X_t^{Ψ}). It is shown that (Q_t) can be approximated, by Trotter-Kato type formula, using (X_t^{Ψ}) and (X_t^{ξ}). Some explicit examples are given. It is shown that in the case of Markov chain (ξ_t), the local behaviour of (X_t) can be obtained, by means of Cameron-Martin formula, from (X_t^{Ψ}). Finally, for deterministic (ξ_t) the explicit formulas for (X_t) are obtained, which show that "the mass" of (X_t) is governed by Ψ, while the "spatial movement" of (X_t) is governed by (ξ_t). In section 3.2 a Zero-One law for Brownian motion is obtained, which complements Engelbert-Schmidt Zero-One law.

CHAPTER 1
INTRODUCTION

Discrete-valued branching processes were studied in the 19th century as mathematical models in a discussion of the survival of family names. In this century applications broadened toward studies of the survival of mutant genes, nuclear chain reactions, and many others. Considering the problem of population growth, it was suggested by William Feller that in the case of large populations continuous models work better than discrete ones (which are succesful in the case of relatively small populations). He presented his programme to the mathematical public in 1951 [26]. In this paper Feller introduced a class of diffusion processes, with values in the set of positive real numbers, which can be obtained as limits of discrete-valued branching processes. Feller also computed the transition probabilities of these diffusions and characterized them in terms of corresponding differential equations.

The consequences of Feller's work [26] are extensive, and we will not attempt to follow all of them. However, two lines of developments coming from Feller's work [26] are important to us. We will see that this two lines later merged.

The first of these two lines, and rather central to us (although it appeared later than the second one), started with the pioneering work [8] of Donald Dawson in 1975. Dawson followed the line of extension of Feller's consideration to the space-time context. Namely, although Feller's diffusions are continuum valued, they are models for the number of individuals (or total mass) of a population. The idea was later to find models for populations distributed in space, which are changing through time. Such processes will have to be measure-valued, and, as it was already noticed

before 1975, will be described by *nonlinear* stochastic evolution equation. Dawson's idea was to consider n individuals of equal mass located at n points $y_1, \ldots, y_n$ in the space $E \subseteq \mathbb{R}^d$. Each individual has an exponentially distributed lifetime with mean $1/\gamma$, and at the end of its lifetime it is replaced by r similar offspring with probability p_r, all initially located at the final position of their parent. Each individual moves independently through space E, according to the given law of a Markov process (ξ_t), whose infinitesimal generator we denote by A. If $N(t)$ is the total number of individuals at the moment t, then the mathematical model for the situation above is the branching diffusion

$$X_t^n(\omega) = \sum_{i=1}^{N(t,\omega)} \delta_{y_i(\omega)} \quad , \tag{1.1}$$

where δ_y is the point-mass at $y \in E$. The continuous analog of (X_t^n) led Dawson to the stochastic evolution equation

$$dX(t) = (A + \alpha)X(t)\,dt + \sqrt{2\gamma X(t)}dB(t) \quad , \tag{1.2}$$

where $B(t)$ is, so called, Brownian motion with spatially correlated increments. The main technical difficulty, in giving a mathematical meaning to equation 1.2, is the appearance of the non-Lipschitz factor $\sqrt{2\gamma X(t)}$. By treating the characteristic functionals of (X_t^n) through the application of semigroup theory, Dawson succeded in defining rigorously measure-valued process (X_t), which can be understood as the solution of 1.2. He also showed that, by properly adjusting the parameters, (X_t) is the limiting case of branching diffusions (X_t^n) as $n \longrightarrow +\infty$. It was done in the case when (ξ_t) is a standard Brownian motion, and $p_0 = 1/2 - 2\alpha/n$, $p_2 = 1/2 + 2\alpha/n$. The process (X_t) is the example of what was later to be named superprocess. In doing so Dawson not only offered one of the main ideas for construction of superprocesses, but also gave the intuitive meaning to (X_t) as the mathematical model for

the propagation of a group of individuals (or cloud of particles, for example) through space and time.

In the succeeding few years Dawson outlined main directions in the study of multiplicative (i.e. branching) stochastic measure diffusion processes, as he called them then, or superprocesses, as most of the authors call them today. His programme was carried through most of the eighties. At the end of the eighties and the beginning of the nineties several new ideas emerged, but we will say more about it later.

Except for the paper [8], two other papers in the late seventies were crucial for the development of the theory of superprocesses. In 1977 Dawson, in his paper [9], studied the limiting behaviour of superprocesses, i.e., the behaviour of X_t as $t \longrightarrow +\infty$. In particular, the questions of extinction and of the existence of invariant distribution were studied. This line of study was continued and developed [11], [19], [32]. I. Iscoe [34] introduced weighted occupation time processes and investigated their behaviour when $t \longrightarrow +\infty$.

Another important early paper was published in 1979 by Dawson and Hochberg [12]. In this paper local properties were studied, i.e., if we fix $t > 0$, then we can ask questions about the nature of the measure X_t. In many cases it is a singular measure, so the questions about the support of X_t naturally came into consideration. This type of questions were further studied by Sylvie Roelly-Coppoletta in her work [52] and other works [58] and [43]. In 1988 Edwin Perkins [48] considered the question of the nature of the random measure over time intervals (i.e. not just for a fixed $t > 0$). Special consideration was given to the case of the superbrownian motion (superprocess over standard Brownian motion (ξ_t)), and to the study of its paths [13]. All this led to the development of an important new idea, i.e., the idea of a historical process. It is a superprocess with values in the space of measures on the space $D(E)$ of E-valued càdlàg paths on $\mathbb{R}_+ = [0, +\infty)$. Intuitively, the historical

process takes into consideration the "genealogy" of the starting superprocess (X_t). For detailed study we refer to Dawson and Perkins [15]. Let us also mention the paper [25] by Steven Evans and E. Perkins in which they introduce the notion of a collision measure. They use it to "gauge" the overlap of a fixed measure and a superprocess.

Throughout all that period one of the main problems was the problem of the construction of superprocesses. Dawson's starting idea, given in 1975, was later formalized in terms of a martingale problem (developed by Stroock and Varadhan). We refer to [14], [52] for these results, and to [24] for general treatment. In [52] the martingale properties of superprocesses were emphasized. In the following years more general processes (ξ_t), more general conditions on branching $\{p_r\}$, and more general initial conditions were considered. For construction of superprocesses on p-tempered distributions see [34], for further formalization in terms of stochastic differential equations see [43], and, together with further study of martingale properties, [46]. Already in the eighties superprocesses (X_t) were characterized as measure-valued processes, such that their Laplace transforms were given by

$$\mathrm{E}\left[\mathrm{e}^{-\int_E f(x)dX_t}\right] = \mathrm{e}^{-\int_E (V_t f)(x)d\mu} \;\; ; \;\; X_0 \equiv \mu \;, \tag{1.3}$$

where $f \geq 0$ is bounded, continuous function on E, and (V_t) is a nonlinear semigroup which satisfies the equation

$$V_t f = T_t f + \int_0^t T_{t-s}[\Psi(V_s f)]ds \;\; , \tag{1.4}$$

where (T_t) is the Markov semigroup of (ξ_t) and $\Psi(\lambda) = -\gamma\lambda^2$. These formulas suggested strong connections with the work on branching processes done by S. Watanabe in 1968. It was further exploited by P. Fitzsimmons and E. Dynkin. But before we go to their work, let us exploit the other line of development related to Feller's work mentioned at the beginning.

In 1951, when Feller offered his programme, the notion of branching processes meant only the discrete-valued processes. Miloslav Jiřina, in his paper 1958 [36], developed the notion of branching property to processes with continuous state space. Since additive structure of state space is needed, measure-valued processes represent an important example. However, it was not known how to characterize continuous-valued processes as branching. Of course, the simplest case was the case of $\mathbb{R}_+ = [0, +\infty)$. Real-valued branching processes were characterized as time-changed Lévy processes by John Lamperti in 1967 [44]. It turned out that $\mathbb{R}_+$-valued branching diffusions are exactly the diffusions obtained by Feller in 1951. The next step, i.e., the characterization of $\mathbb{R}_+^n$-valued branching processes was carried out by Shinzo Watanabe in 1969 [57]. To simplify notation, he worked on the case $n = 2$. Watanabe characterized completely these processes in terms of their infinitesimal generators, and, in the case of diffusions, analyzed the behaviour on the boundary of $\mathbb{R}_+^n$. Independently, same results for infinitesimal generators were obtained by Yu.M. Ryzhov and A.V. Skorokhod in 1970 [51]. More general branching processes with immigration were treated in another work [40]. An account of branching semigroups on the space of finite measures on compact separable metric space was given in 1969 by M.L. Silverstein [55]. Most recently, Dynkin, Kuznetsov, and Skorokhod studied the structure of general branching measure-valued processes [22]. However, the most important work for our consideration is the paper [56] by S. Watanabe, published in 1968. Watanabe constructed the class of measure-valued branching processes (X_t), characterized by 1.3 and 1.4, where E is a separable compact metric space, and Ψ is, so called branching mechanism, given by

$$\Psi(\lambda) = a_0 + a_1\lambda - a_2\lambda^2 - \int_0^{+\infty} \left(e^{-\lambda u} - 1 + \frac{\lambda u}{1+u} \right) n(du) \quad , \tag{1.5}$$

where $\lambda \in (0, +\infty)$, $a_0, a_1, a_2 \in \mathbb{R}$, $a_0, a_2 \geq 0$, and n is a measure on $(0, +\infty)$ such that

$$\int_0^{+\infty} \frac{u^2}{1+u^2} n(du) < +\infty \quad . \tag{1.6}$$

His assumption was that (ξ_t) is a Feller process, and then he proved that (X_t) is also Feller and characterized its infinitesimal generator.

Now we can go back to the story of superprocesses. In 1988 Patrick Fitzsimmons offered a new way of constructing superprocesses, following Watanabe's approach. He considered a very general class of Markov processes (ξ_t), and very general form of Ψ with nonconstant coefficients. He showed [28], [29] that the corresponding, i.e., (ξ, Ψ)-superprocess can be constructed and that it satisfies nice regularity properties. The superprocess is now defined so that it satisfies 1.3 and 1.4. Similar approach was taken by Eugene B. Dynkin [18]. He considered branching mechanism $\Psi(\lambda) = -\lambda^2$, and time nonhomogeneous case. Also, his methods are different from Fitzsimmons'. In this way Watanabe's and Dawson's theories were formally unified. Dynkin is also responsible for the introduction of the term superprocess over Markov process (ξ_t) with branching mechanism Ψ. We will follow this terminology and mostly Fitzsimmons' line of approach in this work. We will actually use the term Dawson-Watanabe superprocess, since there are other types of processes (Fleming-Viot, Ornstein-Uhlenbeck) sometimes called superprocesses, which will not be considered in this work.

Let us finish this brief survey of the theory of superprocesses by mentioning some other recent developments, which otherwise will not be treated in this work. The third, very different, construction of the superbrownian motion was recently offered by Jean-François Le Gall [30]. The connections between the theory of superprocesses and PDE-s has been investigated by Dynkin. The systematic treatment of these problems is given in [20]. Part III of this paper contains a nice historical survey of

the theory of superprocesses. The Lévy-Hinčin represantation for superprocesses is analysed in [39].

Despite all these achievements and thorough analysis of some particular cases, like superbrownian motion, it appears that the "nature" of superprocesses is still somewhat hidden from us. For example, even in the case of Markov chain (ξ_t), where the corresponding superprocess (X_t) is an $\mathbb{R}_+^n$-valued branching process, and we can characterize the infinitesimal generator of (X_t) completely (see [57]), it is still not completely clear which part of the behaviour of (X_t) comes from (ξ_t) and which from Ψ. Intuition would suggest that (ξ_t) influences "spatial motion," while Ψ governes "changing of the mass." Can we describe mechanisms which govern these behaviours? And what about other type of processes, in more general spaces? It seems that one of the main difficulties is in the fact that equation 1.4 is nonlinear, and in general we can not expect to have explicit solutions of 1.4. However, if we concentrate only on "components" of superprocesses we can hope to characterize, through explicit formulas even, these "components" rather completely, and then to use them to recover some information from the most general cases. After establishing superprocesses in a very general situation, we will follow this line of approach, i.e., we will try to give as explicit formulas for the "components" as we can, and then to recover general formulas from these, if possible. We will say what we can in general cases, but we will concentrate on the simplest examples of (ξ_t). Although we can not completely answer the questions mentioned above, we hope that this work offers some improvement and clarification in this direction.

The thesis is organized into five chapters, where the first chapter is this introduction, and the fifth chapter is a conclusion. Hence, the results are contained in chapters 2,3, and 4.

In chapter 2 we deal with the problem of the foundation of superprocesses. We take an approach through the semigroup theory of Markov processes. Hence, we have to solve a nonlinear integral equation first, and then to show that the result is the exponent of the Laplace transform of the transition probability of the superprocess. This method was used by both Watanabe and Fitzsimmons, but the first dealt with the Feller semigroup on a compact state space, while the second dealt with more general Lusin state space. In section 2.1 we offer a unified approach, so that both Watanabe's and Fitzsimmons' result are special cases of our general method, as is shown in sections 2.2 and 2.4. Actually our method solves the problem of positivity preserving nonlinear perturbation of positive semigroups, and the class of semigroups included is very large. Particularly useful is Lemma 2.1.3, which we will apply to deal with linear case and Kac semigroup in section 2.3. In section 2.5 we mention the branching property of superprocesses, and how it simplifies our dealing with the transition probabilities of superprocesses.

In chapters 3 and 4 we analyze the transition probability for particular superprocesses. We start by describing the simplest, one-dimensional superprocesses, which were studied first by Lamperti. We compute some new explicit formulas in these cases. In section 3.2 we also prove a zero-one law for Brownian motion, which complements the Engelbert-Schmidt zero-one law. In section 3.3 we introduce a notion of measures on rays; these are example of measures on the space of measures. Using this notation we can precisely describe transition probabilities for superprocesses over ξ, where $\xi_t \equiv \xi_0$, denoted by (X_t^Ψ). This will be done in section 3.4.

In chapter 4 we use results from the previous chapter to interpret superprocesses in terms of their "component processes" X^Ψ and X^ξ, which are much simpler. X^ξ is obtained when $\Psi = 0$ and is a deterministic process, which is easy to describe. In the

general case we can only approximate the transition probability of X through transition probabilities of the "component processes" (see section 4.1). However, in some special cases we can have more explicit interpretations. The case of deterministic (ξ_t) is rather completely resolved in section 4.3, where we obtain nice explicit formulas. The case of Markov chains is already more complicated and we were able only to describe the local behaviour of X through the local behaviour of X^{Ψ}, by application of Cameron-Martin formula. This will be shown in section 4.2.

CHAPTER 2
GENERAL RESULTS

2.1 Nonlinear Perturbations of Positive Semigroups

In this section we consider the problem of nonlinear perturbation of a linear positive semigroup, in such a way that the positivity of semigroup is preserved. Although our approach relys on standard iteration method and is specially suited for our purposes, we believe that the fact that we consider a very general class of semigroups and nonlinear operators, makes it important even in an abstract framework of mathematical analysis. A good reference for the standard case, i.e., linear perturbation of not necessarily positive and C_0-semigroups is [47], and for the case of perturbation by branching mechanism (see the next section) the references are [56], [28], and [18].

Let $(\mathbf{B}, \| \|, \leq)$ be a Banach space and a linear lattice with closed positive cone. More precisely, $(\mathbf{B}, \| \|)$ is a Banach space and $(\mathbf{B}, \leq)$ is a partially ordered set with the following properties:

$$\text{for every } x, y, z \in \mathbf{B} \quad x \leq y \implies x + z \leq y + z \tag{2.1}$$

$$\text{for every } x, y \in \mathbf{B} \text{ and real } \lambda \geq 0. \quad x \leq y \implies \lambda x \leq \lambda y \tag{2.2}$$

$$\text{for every } x \in \mathbf{B} \text{ there exists } \sup\{x, 0\}, \text{ denoted by } x \vee 0 \text{ or } x^+ \tag{2.3}$$

$$\mathbf{B}^+ = \{x \in \mathbf{B} \mid x \geq 0\} \text{ is closed in the norm topology} \quad . \tag{2.4}$$

It can be shown (see for example [53]) that in $(\mathbf{B}, \leq)$ $x \vee y$ and $x \wedge y$ exist, for every $x, y \in \mathbf{B}$, and that $x \vee y + x \wedge y = x + y$. Then we define x^- to be $(-x) \vee 0$, and $|x|$ to be $x^+ + x^-$. Some of the properties that x^+, x^-, and $|x|$ satisfy are:

$$x = x^+ - x^-, \text{ i.e., } \mathbf{B} = \mathbf{B}^+ - \mathbf{B}^+ \tag{2.5}$$

$$x^+ \vee x^- = x \vee (-x) = |x| \qquad x^+ \wedge x^- = 0 \tag{2.6}$$

$$|x + y| \leq |x| + |y| \qquad |\lambda x| = |\lambda|\,|x| \quad . \tag{2.7}$$

Remark 2.1.1 Notice that $(\mathbf{B}, \| \|, \leq)$, as defined above, is not necessarily a Banach lattice. *Banach lattice* is a partially ordered Banach space which satisfies 2.1-2.3, and the property that, for every $x, y \in \mathbf{B}$,

$$|x| \leq |y| \implies \|x\| \leq \|y\| \quad . \tag{2.8}$$

If $\mathbf{B}$ is a Banach lattice then the mappings $x \mapsto x^+$, $x \mapsto x^-$, $(x,y) \mapsto x \vee y$ are uniformly continuous, and 2.4 is satisfied. See [53], II.5. for details. Therefore, the class of Banach spaces that we consider is bigger than the class of Banach lattices. Typical example of the space which satisfies 2.1-2.4 is a Sobolev space $\mathrm{H}^1((a,b))$ with the relation of partial order defined by ($f \leq g$ if $f(x) \leq g(x)$ for every $x \in (a,b)$). But $\mathrm{H}^1((a,b))$ is not a Banach lattice. It is easy to construct a sequence of nonnegative functions $\{f_n\}$ in $\mathrm{H}^1((a,b))$, such that $0 \leq f_n \leq 1$ and $\|f_n'\|_{L^2} \longrightarrow +\infty$. Thus $\mathrm{H}^1((a,b))$ does not satisfy 2.8.□

As usual, we will say that an operator (not necessarily linear) $O : \mathbf{B} \longrightarrow \mathbf{B}$ is *positive* if $O(\mathbf{B}^+) \subseteq \mathbf{B}^+$.

Let $\mathbf{B}^*$ be the dual space of $\mathbf{B}$, i.e., the space of all bounded linear functionals on $\mathbf{B}$. We define the cone of positive functionals $\mathbf{B}^{*,+}$ by

$$\mathbf{B}^{*,+} = \left\{ f \in \mathbf{B}^* \mid f(x) \geq 0, \text{ for every } x \in \mathbf{B}^+ \right\} \quad . \tag{2.9}$$

It is true that, if $f(x) \geq 0$, for every $f \in \mathbf{B}^{*,+}$, then $x \in \mathbf{B}^+$. This statement follows from the fact that $\mathbf{B}^+$ is closed, see [41] p.226.

We will slightly generalize the notions of weak measurability and weak integration (see the examples below for the cases which require such generalizations). Let $\mathbf{M} \subseteq$

$\mathbf{B}^*$ be a nonempty subset of $\mathbf{B}^*$ with the following properties:

$$(\mathrm{M}, \|\ \|) \text{ is a Banach space, where } \|\ \| \text{ is the operator norm of } \mathbf{B}^* \tag{2.10}$$

$$m(x) = m(y)\,, \text{ for every } m \in \mathrm{M} \implies x = y \tag{2.11}$$

$$m(x) \geq 0\,, \text{ for every } m \in \mathrm{M}^+ = \mathrm{M} \cap \mathbf{B}^{*,+} \implies x \in \mathbf{B}^+ \tag{2.12}$$

$$\|x\| = \sup\left\{ \frac{|m(x)|}{\|m\|}\ ;\ m \in \mathrm{M} \setminus \{0\} \right\}\ . \tag{2.13}$$

Notice that, if $\mathrm{M} = \mathbf{B}^*$, then M satisfies 2.10-2.13. We define σ-algebra $\mathcal{M}$ on M to be the smallest σ-algebra such that, for every $x \in \mathbf{B}$, the mappings $m \mapsto m(x)$ are measurable. Let $I \subseteq \mathbb{R}$ be an interval and $\mathcal{B}(I)$ the σ-algebra of Borel sets on I.

Definition 2.1.1 *A vector-valued function $x(s) : I \longrightarrow \mathbf{B}$ is:*

M-weakly measurable *if $m \circ x$ is $\mathcal{B}(I)/\mathcal{B}(\mathbb{R})$ measurable, for every $m \in \mathrm{M}$.*

M-measurable *if $(m, s) \mapsto m[x(s)]$ is $\mathcal{M} \otimes \mathcal{B}(I)/\mathcal{B}(\mathbb{R})$ measurable.*

M-Pettis integrable *if there exists an element $x_I \in \mathbf{B}$, such that, for every $m \in \mathrm{M}$, $m \circ x$ is Lebesgue integrable and*

$$m(x_I) = \int_I m[x(s)]ds\ . \tag{2.14}$$

In this case x_I is called M-Pettis integral *of $x(s)$ and is denoted by*

$$x_I = (\mathrm{M}P)\int_I x(s)ds\ . \tag{2.15}$$

If $\mathrm{M} = \mathbf{B}^*$, then these are the standard notions of weak measurability and Pettis integrability (see, for example, Chapter III in [33]), and in this case we will denote Pettis integral x_I by

$$(P)\int_I x(s)ds\ . \tag{2.16}$$

The notion of M-measurability requires additional explanation in some cases (see Proposition 2.1.1 and examples below).

Definition 2.1.2 *A linear bounded operator* $B : \mathbf{B} \longrightarrow \mathbf{B}$ *is* M-invariant *if, for every* $m \in \mathrm{M}$, $m \circ B \in \mathrm{M}$.

The basic properties of M-Pettis integrability are given in the following theorem. Notice that $\|x(s)\|$ is not necessarily measurable, even if $x(s)$ is M-Pettis integrable. Various formulations of Fubini theorem are possible. We will state the formulation which serves our purposes.

Theorem 2.1.1 *Let* $x(s)$ *and* $y(s)$ *be* M-*Pettis integrable on* I, *and* α, β *real numbers. Then the following is true:*

$$The\ \mathrm{M}-Pettis\ integral\ of\ x(s)\ is\ unique. \tag{2.17}$$

$$If,\ for\ every\ s \in I,\ x(s) \in \mathbf{B}^+,\ \Longrightarrow\ (\mathrm{M}P)\int_I x(s)ds \in \mathbf{B}^+. \tag{2.18}$$

If $\mathrm{M_1} \subseteq \mathrm{M}$ *and* $\mathrm{M_1}$ *satisfies 2.10-2.13, then* $x(s)$ *is* $\mathrm{M_1}$-*Pettis integrable and two integrals coincide. In particular, if* $x(s)$ *is Bochner or Pettis integrable, then* $x(s)$ *is* M-*Pettis integrable and*

$$(B)\int_I x(s)ds = (P)\int_I x(s)ds = (\mathrm{M}P)\int_I x(s)ds\ , \tag{2.19}$$

where the first integral is Bochner integral.

$$\alpha x(s) + \beta y(s)\ is\ \mathrm{M}-Pettis\ integrable\ and\ the\ integral\ is\ linear. \tag{2.20}$$

If $\alpha(s) : I \longrightarrow [0, +\infty)$ *is Lebesgue integrable and, for every* $s \in I$, $\| x(s) \| \leq \alpha(s)$, *then*

$$\left\| (\mathrm{M}P)\int_I x(s)ds \right\| \leq \int_I \alpha(s)ds \tag{2.21}$$

If B *is* M-*invariant, then* $B[x(s)]$ *is* M-*Pettis integrable and*

$$B\left[(\mathrm{M}P)\int_I x(s)ds\right] = (\mathrm{M}P)\int_I B[x(s)]ds \tag{2.22}$$

If $x(s,t) : I \times I \longrightarrow \mathbf{B}$ *is such that* $s \mapsto x(s,t)$ *and* $t \mapsto x(s,t)$ *are* M*-Pettis integrable, and, for every* $m \in \mathrm{M}$, $m[x(s,t)]$ *is* $\mathcal{B}(I) \otimes \mathcal{B}(I)$ *measurable and integrable on* $I \times I$, *and* $t \mapsto (\mathrm{MP}) \int_I x(s,t)ds$ *is* M*-Pettis integrable, then* $s \mapsto (\mathrm{MP}) \int_I x(s,t)dt$ *is* M*-Pettis integrable and*

$$(\mathrm{MP}) \int_I dt \, (\mathrm{MP}) \int_I ds \, x(s,t) \;=\; (\mathrm{MP}) \int_I ds \, (\mathrm{MP}) \int_I dt \, x(s,t) \; . \tag{2.23}$$

Proof. The statement 2.17 follows from 2.11, 2.18 from 2.12, and 2.19 by definition and the fact that every Bochner integrable function is Pettis integrable and two integrals coincide (see [33], p.80). The statement 2.20 follows directly from the definition, and 2.21 from the definition and 2.13. The statement 2.22 follows from the definition and the fact that, for every $m \in \mathrm{M}$, $m \circ B \in \mathrm{M}$ and (since $x(s)$ is M-Pettis integrable)

$$(m \circ B) \left((\mathrm{MP}) \int_I x(s)ds \right) = \int_I (m \circ B)(x(s))ds \quad .$$

To prove 2.23 notice that there exists a vector $x_0 \in \mathbf{B}$ such that

$$m(x_0) = \int_I dt \; m \left[(\mathrm{MP}) \int_I ds \; x(s,t) \right] = \int_I dt \int_I ds \; m[x(s,t)] \quad .$$

Recall that $m[x(s,t)]$ is integrable on $I \times I$ and real-valued. Thus, by Fubini theorem for real-valued functions,

$$m(x_0) = \int_I ds \int_I dt \; m[x(s,t)] = \int_I ds \; m \left[(\mathrm{MP}) \int_I dt \, x(s,t) \right] \quad ,$$

where the second equality follows from M-Pettis integrability of $t \mapsto x(s,t)$. Notice that $s \mapsto \int_I dt \, m[x(s,t)] \;=\; m[(\mathrm{MP}) \int_I x(s,t)dt \,]$ is Lebesgue integrable and the integral is equal to $m(x_0)$, for every $m \in \mathrm{M}$. By definition, it means that $s \mapsto (\mathrm{MP}) \int_I x(s,t)dt$ is M-Pettis integrable and

$$x_0 = (\mathrm{MP}) \int_I ds \, (\mathrm{MP}) \int_I dt \, x(s,t) \quad .$$

Notice that the consequence of 2.18 and 2.20 is that M-Pettis integral is monotone, i.e., if $x(s)$ and $y(s)$ are M-Pettis integrable, and, for every $s \in I$, $x(s) \leq y(s)$, then

$$(\mathrm{MP}) \int_I x(s)ds \leq (\mathrm{MP}) \int_I y(s)ds \ .$$

Q.E.D.

In the following text we will consider only operators which preserve M-measurability. An operator $O : \mathcal{D}(O) \subseteq \mathbf{B} \longrightarrow \mathbf{B}$ (not necessarily linear) is $\mathrm{M} - measurable$ if $t \mapsto O[x(t)]$ is M-measurable for every M-measurable function $t \mapsto x(t)$ with values in $\mathcal{D}(O)$.

Lemma 2.1.1 *Every* M-*invariant operator* $B : \mathbf{B} \longrightarrow \mathbf{B}$ *is* M-*measurable.*

Proof. Let $x(t) : I \longrightarrow \mathbf{B}$ be M-measurable. It means that the mapping $F(t,m) : I \times \mathrm{M} \longrightarrow \mathbb{R}$, $F(t,m) = m[x(t)]$ is $\mathcal{B}(I) \otimes \mathcal{M}/\mathcal{B}(\mathbb{R})$ measurable. Consider a mapping $H(t,m) : I \times \mathrm{M} \longrightarrow I \times \mathrm{M}$ defined by $H(t,m) = (t,\, m \circ B)$. Notice that $H(t,m) \in I \times \mathrm{M}$, since B is M-invariant. We claim that H is $\mathcal{B}(I) \otimes \mathcal{M}/\mathcal{B}(I) \otimes \mathcal{M}$ measurable. It is enough to prove that $(t,m) \mapsto m \circ B$ is $\mathcal{B}(I) \otimes \mathcal{M}/\mathcal{M}$ measurable, i.e., it is enough to prove that $m \mapsto m \circ B$ is $\mathcal{M}/\mathcal{M}$ measurable. By the definition of $\mathcal{M}$, it is enough to prove that, for every $x \in \mathbf{B}$, $m \mapsto (m \circ B)(x)$ is $\mathcal{M}/\mathcal{B}(\mathbb{R})$ measurable. Since $Bx \in \mathbf{B}$, it is obviously true. Now, $(t,m) \mapsto m[B(x(t))] = F \circ H$ finishes the proof.

Q.E.D.

Let us specify the class of semigroups that we will consider in this section.

Definition 2.1.3 *A semigroup* $(\, T_t \,;\, t \geq 0\,)$, $T_0 = id$, *of bounded linear operators on* $\mathbf{B}$ *is* a positive, M-Pettis integrable semigroup *if it satisfies the following properties:*

$$T_t \text{ is } \mathrm{M} - \text{invariant and positive, for every } t \geq 0 \ . \tag{2.24}$$

There exist $D \geq 1$ *and* $\omega \geq 0$, *such that* $\|T_t\| \leq D\, e^{\omega t}$, *for every* $t \geq 0$. (2.25)

For every $x(t) : (0, +\infty) \longrightarrow \mathbf{B}$, $\mathbf{M}$*-measurable and bounded on bounded subintervals, the mapping* $(m, s, t) \mapsto m[T_s[x(t)]]$ *is* $\mathcal{M} \otimes \mathcal{B}((0, +\infty)) \otimes \mathcal{B}((0, +\infty))/\mathcal{B}(\mathbb{R})$ *measurable and, on every bounded subinterval, the mapping*

$$s \mapsto T_s[x(s)] \quad \textit{is } \mathbf{M} - \textit{Pettis integrable.} \tag{2.26}$$

Conditions 2.24 and 2.25 are standard. Condition 2.26, which is rather technical, enables us to use Fubini-type theorem and weak integration. We will see in examples that 2.26 is fulfilled under more standard assumptions.

Example 2.1.1 Let $(\mathbf{B}, \| \ \|, \leq)$ be a separable Banach space and a linear lattice with closed positive cone. Among such spaces are $L^p(\mu)$, $1 \leq p < +\infty$, where μ is a σ-finite measure defined on a countably generated σ-algebra, $C_0(X)$, where X is a locally compact Hausdorff space with a countable base, and Sobolev spaces $W^{1,p}(I)$, where $1 \leq p < +\infty$. In all these spaces $\leq$ is inherited from the order on the real line. Let $\mathbf{M} = \mathbf{B}^*$. Notice that in this case every bounded linear operator is M-invariant. Let $(T_t;\ t \geq 0)$ be any positive semigroup of bounded linear operators, which satisfies 2.25 and such that $t \mapsto T_t x$ is weakly measurable. Since $\mathbf{B}$ is separable, $t \mapsto T_t x$ is strongly measurable, and, since (T_t) is a semigroup, $t \in (0, +\infty) \mapsto T_t x$ is strongly continuous (see [33], Theorem 10.2.3.). Consider now a mapping $(s, f) \in (0, +\infty) \times \mathbf{B}^* \longrightarrow \mathbf{B}^*$ given by $(s, f) \mapsto f \circ T_s$. For every $x \in \mathbf{B}$, $(s, f) \mapsto f(T_s x)$ is $\mathcal{M}$-measurable in the second coordinate, and continuous in the first coordinate $s \in (0, +\infty)$. Since $(0, +\infty)$ is a separable metric space, $(s, f) \mapsto f(T_s x)$ is $\mathcal{B}((0, +\infty)) \otimes \mathcal{M}/\mathcal{B}(\mathbb{R})$ measurable. Therefore $(s, f) \mapsto f \circ T_s$ is $\mathcal{B}((0, +\infty)) \otimes \mathcal{M}/\mathcal{M}$ measurable. If $x(t)$ is M-measurable and bounded on bounded subintervals, then

$$(f, s, t) \mapsto (f \circ T_s, t) \mapsto (f \circ T_s)(x(t))$$

is $\mathcal{M} \otimes \mathcal{B}((0,+\infty)) \otimes \mathcal{B}((0,+\infty))/\mathcal{B}(\mathbb{R})$ measurable. In particular, $(f,s) \mapsto (f \circ T_s)(x(s))$ is $\mathcal{M} \otimes \mathcal{B}((0,+\infty))/\mathcal{B}(\mathbb{R})$ measurable. Since $\mathbf{M} = \mathbf{B}^*$, $s \mapsto T_s[x(s)]$ is weakly measurable, and $s \mapsto \|T_s[x(s)]\|$ is bounded on bounded subintervals. $\mathbf{B}$ is separable implies that $s \mapsto T_s[x(s)]$ is strongly measurable, and, therefore, Bochner integrable on bounded subintervals. It shows that (T_t) satisfies 2.26.$\diamondsuit$

Example 2.1.2 Let $(E,\mathcal{E})$ be a measurable space, and $(\mathbf{B}, \| \|)$ a Banach space of bounded $\mathcal{E}$-measurable real-valued functions with the sup norm. Then we have a natural partial order on $\mathbf{B}$, inherited from the real line. Such a space $\mathbf{B}$ satisfies 2.1-2.4, and, even more 2.8. Let $\mathbf{M}$ be a set of all finite signed measures on $(E,\mathcal{E})$. Then $\mathbf{M}$ with the total variation norm satisfies 2.10-2.13, and $\mathbf{M} \neq \mathbf{B}^*$. Let $p_t(x,dy)$ be a positive kernel on $E \times \mathcal{E}$, such that $(t,x) \mapsto p_t(x,dy)$ is measurable, and there exist $D \geq 1$, $\omega \geq 0$, such that, for every $t \geq 0$ and $x \in E$, $p_t(x,E) \leq D\exp(\omega t)$. We define T_t on $\mathbf{B}$ by

$$(T_t f)(x) = \int_E f(y)\, p_t(x,dy) \quad . \tag{2.27}$$

Typical example of such a semigroup is a Markov transition function $p_t(x,dy)$ and its semigroup, when it is a transition function of a Borel right process with the state space $(E,\mathcal{B}(E))$, where E is a Lusin topological space and $\mathcal{B}(E)$ the σ-algebra of Borel sets (see [54], pp.13, 105, and App.2). In this case $\omega = 0$ and $D = 1$.

Let us show that (T_t), defined by 2.27, satisfies all the requirements of Definition 2.1.3. Since $p_t(x,dy)$ is a positive kernel and $\|T_t\| = \sup\{p_t(x,E); x \in E\}$, 2.24 and 2.25 follows immediately from our assumptions. Let us denote $\int_E f(x)\mu(dx)$ by $\langle f,\mu\rangle$, where $f \in \mathbf{B}$ and $\mu \in \mathbf{M}$. Consider a function $F(s,x) : (0,+\infty) \times E \longrightarrow \mathbb{R}$ defined by $F(s,x) = 1_B(s) \cdot 1_H(x)$, where $B \in \mathcal{B}((0,+\infty))$, $H \in \mathcal{E}$. Then $\langle F(s,\cdot),\mu\rangle = 1_B(s) \cdot \mu(H)$, and, therefore, $(s,\mu) \mapsto \langle F(s,\cdot),\mu\rangle$ is $\mathcal{B}((0,+\infty)) \otimes \mathcal{M}/\mathcal{B}(\mathbb{R})$ measurable. Using the monotone class argument, it follows that $(s,\mu) \mapsto \langle F(s,\cdot),\mu\rangle$ is

$\mathcal{B}((0,+\infty)) \otimes \mathcal{M}/\mathcal{B}(\mathbb{R})$ measurable for every bounded $\mathcal{B}((0,+\infty)) \otimes \mathcal{E}/\mathcal{B}(\mathbb{R})$ measurable $F(s,x)$. By our assumption, $F(s,x) = (T_s f)(x)$ is such a function, for every $f \in \mathbf{B}$. Hence, for every $H \in \mathcal{E}$ and $B \in \mathcal{B}((0,+\infty))$,

$$\langle 1_B(t) \cdot 1_H(\cdot),\ \mu T_s \rangle = 1_B(t) \cdot \langle T_s(1_H),\ \mu \rangle \quad ,$$

which implies that $(\mu, t, s) \mapsto \langle G(t,\cdot),\ \mu T_s \rangle$ is jointly measurable, for $G(t,y) = 1_H(y) \cdot 1_B(t)$. By monotone class argument, the same is true for any jointly measurable and bounded function $G(t,y)$. Notice that we can identify $(E,\mathcal{E})$ with $(\{\delta_x; x \in E\}, \mathcal{M} \cap \{\delta_x; x \in E\})$, where δ_x is a point mass at x. Therefore, if $f(s) : (0,+\infty) \longrightarrow \mathbf{B}$ is M-measurable, then $(s,x) \mapsto f(s)(x) = \langle f(s), \delta_x \rangle$ is $\mathcal{B}((0,+\infty)) \otimes \mathcal{E}/\mathcal{B}(\mathbb{R})$ measurable and bounded on any bounded subinterval. It follows that

$$(\mu, s, t) \mapsto \langle T_s[f(t)], \mu \rangle = \langle f(t), \mu T_s \rangle$$

is $\mathcal{M} \otimes \mathcal{B}((0,+\infty)) \otimes \mathcal{B}((0,+\infty))/\mathcal{B}(\mathbb{R})$ measurable. In particular, it shows that $(s,x) \mapsto T_s[f(s)](x)$ is $\mathcal{B}((0,+\infty)) \otimes \mathcal{E}/\mathcal{B}(\mathbb{R})$ measurable and bounded on bounded subintervals. It follows, by Fubini theorem, that

$$f_0(x) = \int_{(0,t)} T_s[f(s)](x) ds$$

exists, for every $x \in E$, and defines a bounded $\mathcal{E}$-measurable function f_0. By Fubini theorem, for every $\mu \in \mathrm{M}$,

$$\langle f_0, \mu \rangle = \int_{(0,t)} \langle T_s[f(s)], \mu \rangle ds \ ,$$

which shows that $s \mapsto T_s[f(s)]$ is M-Pettis integrable on every interval $(0,t)$. Hence, (T_t) satisfies all the requirements of Definition 2.1.3.$\diamondsuit$

The main problem of this section is the perturbation of a positive, M-Pettis integrable (linear) semigroup. Let (T_t) be a linear, positive, M-Pettis integrable semigroup (as defined in Definition 2.1.3). Let $L : \mathbf{B}^+ \longrightarrow \mathbf{B}$ be an M-measurable (not

necessarily linear) operator such that $L[x(t)]$ is bounded on bounded subintervals, whenever $x(t) \in \mathbf{B}^+$ is bounded on bounded subintervals.

We would like to solve an integral equation

$$V_t x = T_t x + (\mathrm{M}P) \int_{(0,t)} T_s[L(V_{t-s}x)]ds \quad , \tag{2.28}$$

where $x \in \mathbf{B}^+$. When we say that (V_t) is *the solution of 2.28*, we mean that $(V_t;\ t \geq 0)$ is a family of operators $V_t : \mathbf{B}^+ \longrightarrow \mathbf{B}^+$, such that $t \mapsto V_t x$ is M-measurable and bounded on bounded subintervals, and (V_t) satisfies 2.28. Notice that under these assumptions $s \mapsto T_s[L(V_{t-s}x)]$ is M-Pettis integrable (see 2.26). Thus, 2.28 makes sense.

We will solve the problem by iteration method, which relies on well-known Gronwall's inequality. In the case of superprocesses the method has been used in the works of Watanabe [56], Fitzsimmons [28], and Dynkin [18]. Gronwall's inequality states that if $\alpha : [0, +\infty) \longrightarrow [0, +\infty)$ is a bounded measurable function with the property that, for every $t \geq 0$,

$$\alpha(t) \leq a + b \int_0^t \alpha(s)ds \quad a, b \geq 0 \quad , \tag{2.29}$$

then, for every $t \geq 0$,

$$\alpha(t) \leq a \cdot \mathrm{e}^{bt} \quad . \tag{2.30}$$

In particular, if $a = 0$ then $\alpha \equiv 0$.

The problem is that some of the functions that we will consider are not measurable. However, we can easily adjust the standard proof of the Gronwall's inequality, to obtain the following, say nonmeasurable, version of the Gronwall's inequality.

Lemma 2.1.2 *Let $\alpha : (0, \tau) \longrightarrow [0, c]$ be a bounded mapping, where $0 < \tau \leq +\infty$, $0 < c < +\infty$. If there is a family of measurable mappings $\{\beta^t_\gamma;\ \gamma \in \Gamma,\ t \in (0, \tau)\}$,*

$\beta_\gamma^t : (0,t) \longrightarrow [0,+\infty)$, *such that, for every* $t \in (0,\tau)$,

$$\alpha(t) \leq a + \sup_{\gamma\in\Gamma} \int_0^t \beta_\gamma^t(s)ds \tag{2.31}$$

and, for every $s \in (0,t)$,

$$\beta_\gamma^t(s) \leq b \cdot \alpha(s) \quad , \tag{2.32}$$

where $a \geq 0$, $b \geq 0$ *are constants, then, for every* $t \in (0,\tau)$,

$$\alpha(t) \leq a \cdot e^{bt} \quad . \tag{2.33}$$

Proof. Notice that $\alpha(t)$ is bounded, but is not necessarily measurable. We claim that, for every $n \in \mathbb{N}$,

$$\alpha(t) \leq \sum_{k=0}^{n-1} a\frac{(bt)^k}{k!} + c\frac{(bt)^n}{n!} \quad , \tag{2.34}$$

and 2.33 follows then directly.

$$\alpha(t) \leq a + \sup_{\gamma_1} \int_0^t \beta_{\gamma_1}^t(s_1)ds_1$$

$$\beta_{\gamma_1}^t(s_1) \leq b \cdot \alpha(s_1) \leq ab + \sup_{\gamma_2} \int_0^{s_1} b\beta_{\gamma_2}^{s_1}(s_2)ds_2$$

$$\vdots \qquad \vdots \qquad \cdot \qquad \vdots$$

$$\beta_{\gamma_{n-1}}^{s_{n-2}}(s_{n-1}) \leq b \cdot \alpha(s_{n-1}) \leq ab + \sup_{\gamma_n} \int_0^{s_{n-1}} b\beta_{\gamma_n}^{s_{n-1}}(s_n)ds_n$$

Hence, $\beta_{\gamma_{n-1}}^{s_{n-2}}(s_{n-1}) \leq ab + \int_0^{s_{n-1}} b^2 c ds_n$, and, going backwards with these inequalities, we obtain

$$\alpha(t) \leq a + \int_0^t ab\,ds_1 + \int_0^t \int_0^{s_1} ab^2 ds_2 ds_1 + \ldots +$$
$$+ \int_0^t \cdots \int_0^{s_{n-1}} b^n c\,ds_n \ldots ds_1 \quad ,$$

which implies 2.34.

Q.E.D.

Let us give sufficient conditions on L, such that the solution of 2.28 exists, is bounded and unique, and has a semigroup property. We will start with boundedness.

Theorem 2.1.2 (Boundedness) If $(V_t; t \geq 0)$ is a solution of 2.28 and L satisfies one of the following two conditions:

a) there exists the smallest nonnegative number, say $\|L\|$, such that, for every $x \in \mathbf{B}^+$,

$$\|Lx\| \leq \|L\| \, \|x\| \quad . \tag{2.35}$$

b) $\mathbf{B}$ is a Banach lattice, $x \mapsto x^+$ is M-measurable operator, and there exists the smallest nonnegative number, say $\|L^+\|$, such that, for every $x \in \mathbf{B}^+$,

$$\|(Lx)^+\| \leq \|L^+\| \, \|x\| \quad . \tag{2.36}$$

then there is a constant $\kappa \geq 0$, such that, for every $x \in \mathbf{B}^+$ and $t \geq 0$,

$$\|V_t x\| \leq D\mathrm{e}^{\kappa t} \cdot \|x\| \quad . \tag{2.37}$$

In the case a) $\kappa = \omega + \|L\|D$, and in the case b) $\kappa = \omega + \|L^+\|D$, where ω and D are defined by 2.25.

Proof. Let us prove the case b). The case a) is analogous, but simpler than the case b). Notice first, that, since $|x^+| = x^+ \leq x^+ + x^- = |x|$, the operator $x \mapsto (Lx)^+$ is M-measurable and $(L[x(t)])^+$ is bounded on bounded subintervals, whenever $x(t) \in \mathbf{B}^+$ is bounded on bounded subintervals. Since (V_t) is the solution of 2.28 it follows that $s \mapsto T_s[(L(V_{t-s}x))^+]$ and $s \mapsto T_{t-s}[(L(V_s x))^+]$ are M-Pettis integrable and that, for every $x \in \mathbf{B}^+$,

$$0 \leq V_t x = T_t x + (\mathbf{M}P) \int_{(0,t)} T_{t-s}[L(V_s x)]ds =$$

$$= T_t x + (\mathbf{MP}) \int_{(0,t)} T_{t-s}[(L(V_s x))^+]ds - (\mathbf{MP}) \int_{(0,t)} T_{t-s}[(L(V_s x))^-]ds$$

Since, for every $s \in (0,t)$, T_{t-s} is positive, we obtain

$$0 \leq V_t x \leq T_t x + (\mathbf{MP}) \int_{(0,t)} T_{t-s}[(L(V_s x))^+]ds \quad .$$

$\mathbf{B}$ is a Banach lattice, thus, by 2.8, we obtain

$$\|V_t x\| \leq \|T_t x\| + \left\| (\mathbf{MP}) \int_{(0,t)} T_{t-s}[(L(V_s x))^+]ds \right\| \quad . \tag{2.38}$$

Let $0 < \tau < +\infty$ be fixed. By assumption $t \in (0,\tau) \mapsto \|V_t x\|$ is bounded. Let $\alpha(t) = \exp(-\omega t)\|V_t x\|$. Then $\alpha : (0,\tau) \longrightarrow [0,+\infty)$ is bounded, and, by 2.25 and 2.38,

$$\alpha(t) \leq D\|x\| + \sup_{m \in \mathbf{M}\setminus\{0\}} \frac{\left| m\left\{ (\mathbf{MP}) \int_{(0,t)} T_{t-s}[(L(V_s x))^+]ds \right\} \right|}{\|m\| e^{\omega t}} \leq$$

$$\leq D\|x\| + \sup_{m \in \mathbf{M}\setminus\{0\}} e^{-\omega t} \int_0^t \frac{|m(T_{t-s}[(L(V_s x))^+])|}{\|m\|} ds \quad .$$

Therefore, we choose

$$\beta_m^t(s) = e^{-\omega t} \frac{|m(T_{t-s}[(L(V_s x))^+])|}{\|m\|} \quad .$$

Then, the conditions of Lemma 2.1.2 are fulfilled, since

$$\beta_m^t(s) \leq D e^{-\omega s} \|(L(V_s x))^+\| \leq D\|L^+\| e^{-\omega s} \|V_s x\| = D\|L^+\| \alpha(s) \quad .$$

Lemma 2.1.2 implies that, for every $t \in (0,\tau)$,

$$e^{-\omega t}\|V_t x\| \leq D\|x\| e^{D\|L^+\| t} \quad .$$

Since τ was arbitrary, $D \geq 1$, and $V_0 x = T_0 x = x$, 2.37 is satisfied for every $t \geq 0$ and $x \in \mathbf{B}^+$.

Q.E.D.

For the existence and uniqueness of the solution we need Lipschitz condition. We will say that $L : \mathbf{B}^+ \longrightarrow \mathbf{B}$ is *locally Lipschitz* if $L(0) = 0$, and, for every $K > 0$, there exists $C(K) > 0$, such that

$$\| Lx - Ly \| \leq C(K) \| x - y \| \tag{2.39}$$

whenever $x, y \in \mathbf{B}^+$, and $\|x\| \leq K$, $\|y\| \leq K$.

Notice that, if $x(t) : I \longrightarrow \mathbf{B}^+$ is bounded, i.e., there exists $K > 0$ such that, for every $t \in I$, $\|x(t)\| \leq K$, then, by 2.39, $L[x(t)]$ is bounded (by $C(K) \cdot K$) on I. Therefore, if L is locally Lipschitz, then it is enough to keep as our starting assumptions (see 2.28) that L is M-measurable. Of course, every locally Lipschitz operator is also continuous.

Theorem 2.1.3 (Uniqueness and semigroup property) If L is locally Lipschitz, then there is no more than one solution of 2.28. If $(V_t; t \geq 0)$ is a solution of 2.28 and L is locally Lipschitz, then (V_t) is a semigroup.

Proof. Assume that there are two solutions, say (V_t) and (U_t), of 2.28. We fix $x \in \mathbf{B}^+$ and $\tau \in (0, +\infty)$. Then there exists $K > 0$, such that $\max\{\|V_t x\|, \|U_t x\|\} \leq K$, for every $t \in [0, \tau]$. By 2.28, 2.20 and linearity of T_s we obtain, for every $t \in [0, \tau]$,

$$V_t x - U_t x = (\mathrm{M}P) \int_{(0,t)} T_s[L(V_{t-s}x) - L(U_{t-s}x)]\, ds =$$

$$= (\mathrm{M}P) \int_{(0,t)} T_{t-s}[L(V_s x) - L(U_s x)]\, ds \quad .$$

It follows that

$$\| V_t x - U_t x \| = \sup_{m \in \mathbf{M}\setminus\{0\}} \frac{\left| \int_0^t (m \circ T_{t-s})[L(V_s x) - L(U_s x)]\, ds \right|}{\|m\|} \leq$$

$$\leq \sup_{m \in \mathbf{M}\setminus\{0\}} \int_0^t \frac{|(m \circ T_{t-s})[L(V_s x) - L(U_s x)]|}{\|m\|}\, ds \quad . \tag{2.40}$$

Consider now $\alpha(t) = \| V_t x - U_t x \|$ on $(0,\tau)$, and $\beta_m^t : (0,t) \longrightarrow [0,+\infty)$, where

$$\beta_m^t(s) = \frac{|(m \circ T_{t-s})[L(V_s x) - L(U_s x)]|}{\|m\|} \quad .$$

By 2.40 we get $\alpha(t) \leq C(K) \cdot 2K \cdot \tau \cdot D \exp(\omega\tau)$, i.e., α is bounded. Since, by Lipschitz condition and 2.25,

$$\beta_m^t(s) \leq \|T_{t-s}\| \cdot \| L(V_s x) - L(U_s x) \| \leq$$

$$\leq D e^{\omega(t-s)} \cdot C(K) \| V_s x - U_s x \| \leq D e^{\omega\tau} \cdot C(K) \cdot \alpha(s) \quad ,$$

Lemma 2.1.2 implies that $\alpha(t) = 0$, for every $t \in (0,\tau)$. Since $V_0 x = U_0 x = x$, and, x and τ were arbitrary, we obtain $V_t = U_t$, for every $t \geq 0$.

A similar idea, with a little more computation, leads to the semigroup property of (V_t). Consider $x \in \mathbf{B}^+$, and $s,t \in [0,+\infty)$. By 2.22 and 2.24, we get, for every $u \in [0,t]$,

$$T_u(V_s x) = T_u \left(T_s x + (\mathrm{MP}) \int_{(0,s)} T_v[L(V_{s-v} x)]\, dv \right) =$$

$$= T_u T_s x + (\mathrm{MP}) \int_{(0,s)} T_{u+v}[L(V_{s-v} x)] dv =$$

$$= T_{u+s} x + (\mathrm{MP}) \int_{(u,u+s)} T_w[L(V_{u+s-w} x)] dw \quad ,$$

where the last equality follows from the fact that it is true for every $m \in \mathbf{M}$. It follows that, for every $u \in [0,t]$,

$$V_{u+s} x - V_u V_s x = (\mathrm{MP}) \int_{(0,u+s)} T_z[L(V_{u+s-z} x)] dz -$$

$$-(\mathrm{MP}) \int_{(u,u+s)} T_w[L(V_{u+s-w} x)] dw - (\mathrm{MP}) \int_{(0,u)} T_y[L(V_{u-y}(V_s x))] dy =$$

$$= (\mathrm{MP}) \int_{(0,u)} T_z[L(V_{u+s-z} x) - L(V_{u-z}(V_s x))] dz =$$

$$= (\mathrm{MP}) \int_{(0,u)} T_{u-w}[L(V_{w+s} x) - L(V_w(V_s x))] dw \quad .$$

We are now in a similar position as in the case of uniqueness. We consider functions $\alpha : (0,t) \longrightarrow [0,+\infty)$, $\alpha(u) = \| V_{u+s}x - V_u(V_sx) \|$, and $\beta_m^u : [0,u] \longrightarrow [0,+\infty)$,

$$\beta_m^u(w) = \frac{\|(m \circ T_{u-w})[L(V_{w+s}x) - L(V_w(V_sx))]\|}{\|m\|} \quad .$$

By using $K = \max\{\max\{\|V_{w+s}x\|,\ \|V_w(V_sx)\|\};\ w \in [0,t]\}$ and Lemma 2.1.2, we obtain, as we did before, that $\alpha(u) \leq C(K) \cdot 2K \cdot t \cdot D \exp(\omega t)$ and $\beta_m^u(w) \leq D \exp(\omega t) \cdot C(K) \cdot \alpha(w)$. Hence, $\alpha \equiv 0$. In particular $V_{t+s}x = V_tV_sx$. Since s, t and x were arbitrary, (V_t) has the semigroup property.

Q.E.D.

Let us show that the perturbed semigroup (V_t) behaves nicely with respect to further linear perturbation. This fact will have an impact on some of our results later, but the special case of this property is needed to prove the existence of the solution of 2.28. Let $(\, S_t\, ;\, t \geq 0\,)$ be a linear, M-Pettis integrable semigroup (it means that (S_t) satisfies all the conditions of Definition 2.1.3 except, maybe, positivity) such that, for every $x \in \mathbf{B}$,

$$S_tx = T_tx + (\mathrm{MP}) \int_{(0,t)} S_s[B(T_{t-s}x)]ds \quad , \tag{2.41}$$

where B is M-invariant bounded linear operator. Recall (Lemma 2.1.1) that such an operator is M-measurable. Thus $L - B$ is M-measurable operator on $\mathbf{B}^+$, such that $(L-B)[x(t)]$ is bounded on bounded subintervals, whenever $x(t) \in \mathbf{B}^+$ is bounded on bounded subintervals. It shows that the following statement makes sense.

Lemma 2.1.3 *If* $(\, V_t;\, t \geq 0\,)$ *is a solution of 2.28, then, for every* $x \in \mathbf{B}^+$ *and* $t \geq 0$,

$$V_tx = S_tx + (\mathrm{MP}) \int_{(0,t)} S_s[(L-B)(V_{t-s}x)]ds \quad . \tag{2.42}$$

Proof. By our assumptions (see 2.26) $s \to S_s[L(V_{t-s}x)]$ is M-Pettis integrable. Thus, we consider the following integrals

$$(\mathrm{MP}) \int_{(0,t)} S_s[L(V_{t-s}x)]ds = (\mathrm{MP}) \int_{(0,t)} T_s[L(V_{t-s}x)]ds +$$

$$+(\mathrm{MP})\int_{(0,t)}\left[(\mathrm{MP})\int_{(0,s)}S_u[B(T_{s-u}(L(V_{t-s}x)))]du\right]\,ds\quad,$$

where equality follows from 2.41. The condition 2.26 guarantees that $(m,u,s)\longrightarrow m\{S_u[B(T_{s-u}(L(V_{t-s}x)))]\}$ is $\mathcal{M}\otimes\mathcal{B}((0,+\infty))\otimes\mathcal{B}((0,+\infty))/\mathcal{B}(\mathbb{R})$ measurable, and is bounded on $(0,t)\times(0,t)$. Hence, the requirements for Fubini theorem 2.23 are fulfilled. Together with the fact that the first integral above appears in 2.28 we obtain

$$(\mathrm{MP})\int_{(0,t)}S_s[L(V_{t-s}x)]ds=V_tx-T_tx+$$

$$+(\mathrm{MP})\int_{(0,t)}\left[(\mathrm{MP})\int_{(u,t)}S_u[B(T_{s-u}(L(V_{t-s}x)))]ds\right]\,du\quad.$$

Notice that $S_u\circ B$ is an M-invariant bounded linear operator. Thus we can apply 2.22 on the integral on the right hand side, to get that it is equal to

$$(\mathrm{MP})\int_{(0,t)}(S_u\circ B)\left[(\mathrm{MP})\int_{(u,t)}T_{s-u}[L(V_{t-s}x)]ds\right]\,du=$$

$$=(\mathrm{MP})\int_{(0,t)}(S_u\circ B)\left[(\mathrm{MP})\int_{(0,t-u)}T_w[L(V_{(t-u)-w}x)]dw\right]\,du=$$

$$=\text{by 2.28}=(\mathrm{MP})\int_{(0,t)}(S_u\circ B)\left[V_{t-u}x-T_{t-u}x\right]\,du=\text{by linearity of } S_u\circ B=$$

$$=(\mathrm{MP})\int_{(0,t)}(S_u\circ B)(V_{t-u}x)du-(\mathrm{MP})\int_{(0,t)}(S_u\circ B)(T_{t-u}x)du=$$

$$=\text{by 2.41}=(\mathrm{MP})\int_{(0,t)}(S_u\circ B)(V_{t-u}x)du-S_tx+T_tx\quad.$$

Finally, it shows that

$$(\mathrm{MP})\int_{(0,t)}S_s[L(V_{t-s}x)]ds=V_tx-T_tx+$$

$$+(\mathrm{MP})\int_{(0,t)}S_u[B(V_{t-u}x)]du-S_tx+T_tx\quad,$$

which, since S_s is linear, finishes the proof.

Q.E.D.

For any $a \in \mathbb{R}$, an operator B, given by $B(x) = ax$, is M-invariant, bounded and linear. Also, $S_t = \mathrm{e}^{at}T_t$ is a linear M-Pettis integrable semigroup, which is also positive. Even more, (S_t) satisfies, for every $x \in \mathbf{B}$,

$$(\mathrm{MP})\int_{(0,t)} \mathrm{e}^{as}T_s[B(T_{t-s}x)]ds = (\mathrm{MP})\int_{(0,t)} a\mathrm{e}^{as}T_t x\, ds =$$

$$= \left(\int_0^t a\mathrm{e}^{as}ds\right) T_t x = \mathrm{e}^{at}T_t x - T_t x = S_t x - T_t x \quad .$$

If we apply Lemma 2.1.3 on a and $-a$, then we obtain the following statement.

Corollary 2.1.1 *A semigroup* $(V_t;\, t \geq 0)$ *is a solution of 2.28 if and only if it is a solution of*

$$W_t x = \mathrm{e}^{-at}T_t x + (\mathrm{MP})\int_{(0,t)} (\mathrm{e}^{-as}T_s)[(L+a)(W_{t-s}x)]ds \ , \tag{2.43}$$

where $a \in \mathbb{R}$.

In the case of superprocesses, Corollary 2.1.1 has been proved, using integration by parts, in [10], p.58.

We will use Corollary 2.1.1 to prove the positivity of the solution of 2.28. To prove the existence of the solution of 2.28, we require several properties on L. We will assign a special name to such L.

Definition 2.1.4 *An operator* $L : \mathbf{B}^+ \longrightarrow \mathbf{B}$ *is a* SOLP-operator *(solvable with positive solution) if it is an* M*-measurable, locally Lipschitz operator which satisfies either condition a) or condition b) from Theorem 2.1.2, and such that, for every* $K > 0$, *there exists* $a = a(K) > 0$, *such that*

$$Lx + ax \in \mathbf{B}^+ \tag{2.44}$$

whenever $x \in \mathbf{B}^+$ *and* $\|x\| \leq K$.

Theorem 2.1.4 *If* $(T_t;\ t \geq 0)$ *is a positive,* M*-Pettis integrable semigroup, and* L *is a* SOLP*-operator, then there is one and only one solution to 2.28. The solution is a semigroup and satisfies boundedness condition 2.37.*

Proof. It is enough to prove the existence of the solution of 2.28. We will prove it under the assumption b) from Theorem 2.1.2. The case of condition a) is analogous, but simpler than the case b).

Without loss of generality we can assume that $D = 1$. Recall that there is a norm $\| \ \|'$ on $\mathbf{B}$, defined by

$$\|x\|' = \sup_{t \geq 0} \|e^{-\omega t} T_t x\| \quad ,$$

such that $\|x\| \leq \|x\|' \leq D\|x\|$, i.e., $\| \ \|$ and $\| \ \|'$ are equivalent norms. It follows that $\|T_t\|' \leq \exp(\omega t)$, and, for every $m \in \mathrm{M}$, $\|m\|' \leq \|m\| \leq D\|m\|'$. All the requirements in this section are of topological or measure-theoretical nature, except for 2.8 and 2.13. Thus, they will not change when we change to the equivalent norm. Also, we will use 2.8 only in case $0 \leq x \leq y$. Therefore, to prove that we can restrict ourselves to the case $D = 1$, we have to prove that $\| \ \|'$ satisfies 2.8 for $0 \leq x \leq y$, and 2.13. If $0 \leq x \leq y$ then, for every $t \geq 0$, $0 \leq \exp(-\omega t) T_t x \leq \exp(-\omega t) T_t y$, since T_t is positive. By 2.8 for the norm $\| \ \|$ we obtain $\| \exp(-\omega t) T_t x\| \leq \| \exp(-\omega t) T_t y\|$, which implies that $\|x\|' \leq \|y\|'$. 2.13 is equivalent to

$$\|x\|' = \sup\{|m(x)|;\ m \in \mathrm{M},\ \|m\|' \leq 1\} \quad .$$

Notice that $\|x\|' \geq \sup\{|m(x)|;\ \|m\|' \leq 1\}$, since $|m(x)| \leq \|m\|' \|x\|'$, for every bounded linear functional. Using 2.13 for $\| \ \|$ we obtain

$$\|x\|' = \sup_{t \geq 0} \|e^{-\omega t} T_t x\| = \sup_{t \geq 0} \sup_{\|m\| \leq 1} |e^{-\omega t} (m \circ T_t)(x)| \ .$$

For every $t \geq 0$ and $m \in \mathrm{M}$, $\|m\| \leq 1$, $h = \exp(-\omega t)(m \circ T_t)$ is in M, since T_t is M-invariant and M is a vector space. Using the equivalence relations for the operator

norms $\|\ \|$ and $\|\ \|'$ we get

$$\|h\|' \leq \mathrm{e}^{-\omega t}\|m\|' \cdot \|T_t\|' \leq \mathrm{e}^{-\omega t} \cdot 1 \cdot \mathrm{e}^{\omega t} = 1 \quad .$$

Hence

$$\|x\|' \leq \sup\{|h(x)|;\ h \in \mathbf{M},\ \|h\|' \leq 1\} \quad ,$$

which proves 2.13 for $\|\ \|'$. Therefore, we can restrict our proof to the case $D = 1$.

Fix a strictly positive number τ and vector $x \in \mathbf{B}^+$. Let K be equal to

$$\|x\| \cdot \mathrm{e}^{(\omega + \|L^+\|)\tau} \quad ,$$

where $\|L^+\|$ is defined by 2.36. Let $a = a(K)$, defined by 2.44. By Corollary 2.1.1 it is enough to prove the existence of the solution of 2.43. We define inductively $V_t^0 x \equiv 0$, and, for every $k \in \mathbb{N} \cup \{0\}$,

$$V_t^{k+1}x = \mathrm{e}^{-at}T_t x + (\mathrm{M}P)\int_{(0,t)} \mathrm{e}^{-as}T_s[(L+a)V_{t-s}^k x]ds \quad . \tag{2.45}$$

We claim that, for every $k \in \mathbb{N} \cup \{0\}$, and, for every $t \in [0,\tau]$, $V_t^k x$ is well-defined, $0 \leq V_t^k x$, $\|V_t^k x\| \leq \|x\| \cdot \exp((\omega + \|L^+\|)t) \leq K$, and $[0,\tau] \ni t \longrightarrow V_t^k x$ is M-measurable. Obviously, for $k = 0$ the statement is correct. Assume that the statement is true for k. Since $L + a$ is M-measurable and locally Lipschitz, it follows that

$$(m,t,s) \longmapsto m[\mathrm{e}^{-as}T_s[(L+a)V_{t-s}^k x]]$$

is $\mathcal{M} \otimes \mathcal{B}([0,\tau]) \otimes \mathcal{B}([0,t])/\mathcal{B}(\mathbb{R})$ measurable, and M-Pettis integrable with respect to s. Therefore, the M-Pettis integral in 2.45 exists and $V_t^{k+1}x$ is well-defined. By Fubini theorem

$$(m,t) \longmapsto m[V_t^{k+1}x]$$

is $\mathcal{M} \otimes \mathcal{B}([0,\tau])/\mathcal{B}(\mathbb{R})$ measurable. Since $\|V_{t-s}^k x\| \leq K$, and $V_{t-s}^k x \geq 0$, it follows, by 2.44, that $(L+a)(V_{t-s}^k x) \geq 0$. By 2.18 and the positivity of T_s we obtain $V_{t-s}^{k+1}x \geq 0$.

Using the positivity of T_s one more time we obtain

$$0 \leq V_t^{k+1}x \leq \mathrm{e}^{-at}T_t x + (\mathrm{M}P)\int_{(0,t)} \mathrm{e}^{-as}T_s[(LV_{t-s}^k x)^+ + aV_{t-s}^k x]ds \ .$$

By 2.8 for positive vectors we obtain

$$\|V_t^{k+1}x\| \leq \mathrm{e}^{-at}\|T_t x\| + \|(\mathrm{M}P)\int_{(0,t)} \mathrm{e}^{-as}T_s[(LV_{t-s}^k x)^+ + aV_{t-s}^k x]ds\| \ .$$

First, we estimate $\|\mathrm{e}^{-as}T_s[(LV_{t-s}^k x)^+ + aV_{t-s}^k x\|$ to be less or equal to $\mathrm{e}^{-as}\cdot\mathrm{e}^{\omega s}(\|L^+\| + a)\|x\|\exp[(\omega + \|L^+\|)(t-s)]$, and then apply 2.21 and 2.25 with $D = 1$, to obtain

$$\|V_t^{k+1}x\| \leq \|x\|\cdot$$

$$\cdot\left(\mathrm{e}^{(\omega-a)t} + \mathrm{e}^{(\omega+\|L^+\|)t}\int_0^t (\|L^+\| + a)\mathrm{e}^{-s(\|L^+\|+a)}ds\right) =$$

$$= \|x\|\left[\mathrm{e}^{(\omega-a)t} + \mathrm{e}^{(\omega+\|L^+\|)t}\cdot(-\mathrm{e}^{-s(\|L^+\|+a)}|_0^t)\right] =$$

$$= \|x\|\mathrm{e}^{(\omega+\|L^+\|)t} \ .$$

By mathematical induction, the statement is true for every $k \geq 0$.

Let $C = C(K) + a$, where $C(K)$ is a Lipschitz constant defined by 2.39. By 2.45, we obtain

$$\| V_t^{k+1}x - V_t^k x \| =$$

$$= \left\|(\mathrm{M}P)\int_{(0,t)} \mathrm{e}^{-as}T_s[(L+a)V_{t-s}^k x - (L+a)V_{t-s}^{k-1}x]ds\right\| \ .$$

The norm

$$\left\|\mathrm{e}^{-as}T_s[(L+a)V_{t-s}^k x - (L+a)V_{t-s}^{k-1}x]\right\| \leq$$

$$\leq \mathrm{e}^{(\omega-a)s}\cdot C\cdot\| V_{t-s}^k x - V_{t-s}^{k-1}x\| \ ,$$

since all the vectors involved are bounded by K, and L is locally Lipschitz. Let K_1 be equal to $\exp(|\omega - a|\tau)$. Applying 2.21 recursively k times we obtain

$$\| V_t^{k+1}x - V_t^k x\| \leq K\cdot K_1^k\cdot C^k.$$

$$\cdot \int_0^t ds_1 \int_0^{t-s_1} ds_2 \cdots \int_0^{t-s_1-\ldots-s_{k-1}} ds_k = K \cdot K_1^k \cdot C^k \cdot \frac{t^k}{k!} \quad .$$

Therefore $\{ V_t^k;\ k \geq 0 \}$ is a Cauchy sequence in $\mathbf{B}^+ \subseteq \mathbf{B}$. $\mathbf{B}^+$ is a closed subset of a Banach space $\mathbf{B}$, thus there exists a limit (in the norm sense)

$$V_t x = \lim_{k \to \infty} V_t^k x \in \mathbf{B}^+ \quad .$$

Since $\|V_t^k x\| \leq K$, and $m[V_t x] = \lim_{k \to \infty} m[V_t^k x]$, it follows that $t \mapsto V_t x$ is M-measurable and bounded on $[0, \tau]$. In particular, it shows that $s \mapsto \mathrm{e}^{-as} T_s[(L + a)V_{t-s}x]$ is M-Pettis integrable, and, for every $m \in \mathbf{M}$,

$$m\left[(\mathrm{M}P)\int_{(0,t)} \mathrm{e}^{-as} T_s[(L+a)V_{t-s}x]ds\right] =$$

$$= \int_0^t m\left[\mathrm{e}^{-as} T_s[(L+a)V_{t-s}x]\right] ds \quad .$$

By the dominated convergence theorem, the integral on the right hand side is the limit of

$$\int_0^t m\left[\mathrm{e}^{-as} T_s[(L+a)V_{t-s}^k x]\right] ds \quad ,$$

for every $m \in \mathbf{M}$. Since $\{V_t^{k+1}x - \mathrm{e}^{-at}T_t x\}$ converges strongly to $V_t x - \mathrm{e}^{-at}T_t x$, and $\mathbf{M}$ separates the points in $\mathbf{B}$, it follows that $V_t x$ satisfies 2.28 for $\exp(-at)T_t$ and $L + a$. By Corollary 2.1.1 $t \mapsto V_t x$ is the solution of 2.28 on $[0, \tau]$, for $x \in \mathbf{B}^+$. Since the uniqueness theorem (Theorem 2.1.3) is valid for every $x \in \mathbf{B}^+$ and every $0 < \tau < +\infty$, we can define $V_t x$ for every $x \in \mathbf{B}^+$ and every $t \geq 0$, so that $V_t x$ satisfies 2.28, $V_t x \in \mathbf{B}^+$, $t \mapsto V_t x$ is M-measurable and bounded on bounded subintervals.

Q.E.D.

Notice that if the condition a) from Theorem 2.1.2 is satisfied, and condition 2.44 is not necessarily satisfied, the same proof as above will show the existence of the solution of 2.28, even when (T_t) is not positive. However, in such a case we can not obtain the positive solution, in general.

Let $(T_t;\ t \geq 0)$ be an M-Pettis integrable semigroup, i.e., (T_t) satisfies conditions in Definition 2.1.3, except positivity. Let $B : \mathbf{B} \longrightarrow \mathbf{B}$ be an M-invariant bounded linear operator. Then B restricted on $\mathbf{B}^+$ is locally Lipschitz (actually Lipschitz), M-measurable, and satisfies the condition a) from Theorem 2.1.2. It implies, as we just said in the paragraph above, that there is a solution (S_t) of 2.28, which is the limit of approximations (S_t^k), given by 2.45, for $a = 0$ (and therefore Corollary 2.1.1 is not needed in this case) and $L = B$, but (S_t) is not necessarily positive. Since all the operators involved are linear, then S_t^k is linear, and therefore S_t is linear, too. Thus, it can be extended on $\mathbf{B} = \mathbf{B}^+ - \mathbf{B}^+$. It shows that there exists a semigroup $(S_t;\ t \geq 0)$ of linear operators such that, for every $x \in \mathbf{B}$, $(m, t) \mapsto m[S_t x]$ is $\mathcal{M} \otimes \mathcal{B}(\mathbb{R}_+)/\mathcal{B}(\mathbb{R})$ measurable, $\|S_t\| \leq D \exp((\omega + \|B\|)t)$, and,

$$S_t x = T_t x + (\mathbf{M}P) \int_{(0,t)} T_s[B(S_{t-s}x)]ds \quad . \tag{2.46}$$

Also, $S_t x = \lim_{k \to \infty} S_t^k x$, where $S_t^0 \equiv 0$, and

$$S_t^{k+1} x = T_t x + (\mathbf{M}P) \int_{(0,t)} T_s[B(S_{t-s}^k x)]ds \quad . \tag{2.47}$$

Let us restrict our attention to the special case, when $\mathbf{B}$ is separable and $\mathbf{M} = \mathbf{B}^*$. As we have seen in Example 2.1.1, in such a case all the functions involved are actually strongly measurable, and, therefore, Bochner integrable. In particular, our semigroup (S_t), given by 2.46, is a semigroup of bounded linear operators which satisfies 2.25 and $t \mapsto S_t x$ is weakly measurable. As we have shown in Example 2.1.1, it implies that (S_t) satisfies all the requirements of Definition 2.1.3, except, maybe, positivity. Therefore, there is a solution $(W_t;\ t \geq 0)$ of the equation

$$W_t x = S_t x + \int_{(0,t)} S_s[(-B)(W_{t-s}x)]ds \quad , \tag{2.48}$$

where the integral involved is the Bochner integral. If we rewrite 2.46, using Bochner integral and linearity of the operators involved, we obtain that

$$T_t x = S_t x + \int_{(0,t)} T_s[(-B)(S_{t-s}x)]ds \quad .$$

These are exactly the conditions of Lemma 2.1.3. Since $(-B) - (-B) = 0$, we get, for every $x \in \mathbf{B}^+$, $W_t x = T_t x$, and, because of linearity, it follows $W_t = T_t$. It shows that

$$S_t x = T_t x + \int_{(0,t)} S_s[B(T_{t-s}x)]ds \quad ,$$

which is exactly 2.41. Hence, we just proved the following statement:

Corollary 2.1.2 *If* $\mathbf{B}$ *is separable and* $(T_t;\ t \geq 0)$ *a semigroup of bounded linear operators on* $\mathbf{B}$*, which satisfies 2.25, and* $t \mapsto T_t x$ *is weakly measurable, then, for every bounded linear operator* $B : \mathbf{B} \longrightarrow \mathbf{B}$*, there exists a unique semigroup* $(S_t;\ t \geq 0)$ *of bounded linear operators on* $\mathbf{B}$*, such that* $t \mapsto S_t x$ *is weakly measurable, and, for every* $t \geq 0$*,*

$$\|S_t\| \leq D \cdot e^{(\omega + \|B\|)t} \quad , \tag{2.49}$$

and (S_t) *satisfies, for every* $x \in \mathbf{B}$*,*

$$S_t x = T_t x + \int_{(0,t)} S_s[B(T_{t-s}x)]ds =$$

$$= T_t x + \int_{(0,t)} T_s[B(S_{t-s}x)]ds \quad , \tag{2.50}$$

where all the integrals involved are Bochner integrals. Moreover, if (T_t) *is also positive, and* B *satisfies 2.44, then* $(S_t;\ t \geq 0)$ *is positive.*

Let us show that some of the measurability conditions given in this section are much simpler in the separable case, then in general case. Recall that in the separable case the σ-algebra of Borel sets with respect to the norm topology is equal to the

smallest σ-algebra on $\mathbf{B}$ such that all mappings $x \mapsto f(x)$, $f \in \mathbf{B}^*$, are measurable. Consider a mapping $(x, f) \mapsto f(x)$ on $\mathbf{B} \times \mathbf{B}^*$. It is continuous with respect to the first variable, and $\mathcal{M}$ measurable with respect to the second variable. Since $\mathbf{B}$ is separable metric, the mapping $(x, f) \mapsto f(x)$ is $\mathcal{B}(\mathbf{B}) \otimes \mathcal{M}/\mathcal{B}(\mathbb{R})$ measurable. This fact enables us to prove the following proposition.

Proposition 2.1.1 *Let* $\mathbf{B}$ *be a separable Banach space and* $\mathbf{M} = \mathbf{B}^*$. *Then a vector-valued function* $t \mapsto x(t)$ *is* $\mathbf{M}$*-measurable if and only if it is weakly measurable. If* $L : \mathcal{D}(L) \subseteq \mathbf{B} \longrightarrow \mathbf{B}$ *is a continuous operator then it is* $\mathbf{M}$*-measurable. In particular, every locally Lipschitz operator is* $\mathbf{M}$*-measurable, and, if* $\mathbf{B}$ *is a Banach lattice,* $x \mapsto x^+$ *is* $\mathbf{M}$*-measurable.*

Proof. If $t \mapsto x(t)$ is weakly measurable then it is $\mathcal{B}(I)/\mathcal{B}(\mathbf{B})$ measurable. Then $(t, f) \mapsto (x(t), f)$ is $\mathcal{B}(I) \otimes \mathcal{M}/\mathcal{B}(\mathbf{B}) \otimes \mathcal{M}$ measurable. Finally, $(t, f) \mapsto (x(t), f) \mapsto f(x(t))$ is a composition of measurable functions, i.e., it is $\mathcal{B}(I) \otimes \mathcal{M}/\mathcal{B}(\mathbb{R})$ measurable. The reverse statement is always true.

If $t \mapsto x(t) \in \mathcal{D}(L)$ is $\mathbf{M}$-measurable, then it is weakly measurable, i.e., it is $\mathcal{B}(I)/\mathcal{D}(L) \cap \mathcal{B}(\mathbf{B})$ measurable. Since L is continuous, it is $\mathcal{D}(L) \cap \mathcal{B}(\mathbf{B})/\mathcal{B}(\mathbf{B})$ measurable, which shows that $t \mapsto L(x(t))$ is weakly measurable, and, therefore, $\mathbf{M}$-measurable. The last statement follows since every locally Lipschitz operator is continuous, and $x \mapsto x^+$ is continuous in Banach lattice.

Q.E.D.

We will finish this section with a remark on condition 2.44. As we have seen it is a sufficient condition for the positivity of a perturbed semigroup. Notice that for some classes of operators this is also a necessary condition. More precisely, let $\mathbf{B} = \mathbb{R}^n$, where $x = (x_1, \ldots, x_n) \leq y = (y_1, \ldots, y_n)$ if $x_i \leq y_i$, for $i = 1, \ldots, n$. Let $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear operator, i.e., A is a matrix $(a_{ij})_{i,j=1}^n$. Then there exists a

uniformly continuous semigroup

$$S_t = e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} \quad ,$$

which satisfies the equality

$$S_t x = x + \int_0^t A(S_{t-s}x)ds \quad .$$

We claim that the following three statements are equivalent:

i) S_t is positive, for every $t \geq 0$

ii) A satisfies 2.44

iii) for every $i, j \in \{1, \ldots, n\}$, $i \neq j$, $a_{ij} \geq 0$.

By Theorem 2.1.4, ii) implies i). iii) implies ii), since $a = \max_i\{|a_{ii}|\}$ will give $Ax + ax \geq 0$, for every $x \geq 0$. Let us prove that i) implies iii). Suppose contrary to the claim that there exist i_0 and j_0 in $\{1, \ldots, n\}$ such that $i_0 \neq j_0$ and $a_{i_0 j_0} < 0$. We consider $x = (x_1, \ldots, x_n)$ such that $x_i = 0$, if $i \neq j_0$, and $x_{j_0} = 1/(-a_{i_0 j_0})$. Then $x_{i_0} = 0$ and $(Ax)_{i_0} = -1$. We define $f(t) = (S_t x)_{i_0}$. Then $f(t)$ is a differentiable function, $f(0) = x_{i_0} = 0$, $f'(0) = (Ax)_{i_0} = -1$, and $f'(t)$ is continuous. Hence, there is $\varepsilon > 0$ such that on $[0, \varepsilon]$ $f'(t) < 0$, and, since $f(0) = 0$, we must have $f(t) < 0$ on $[0, \varepsilon]$. This contradicts i). Hence, the proof is complete and all three statements are equivalent.

2.2 Supersemigroups and Superprocesses

In this section we will define the main notion of this work. We will follow the approach developed by P. Fitzsimmons and E.B. Dynkin, and show that their results are special cases of results from section 2.1.

Let $(\mathbf{B}, \| \ \|, \leq)$ be a Banach space and a linear lattice with norm-closed positive cone. Let $\mathrm{M} \subseteq \mathbf{B}^*$ satisfies 2.10-2.13, and consider a measurable space $(\mathrm{M}^+, \mathcal{M}^+)$, where $\mathrm{M}^+ = \mathrm{M} \cap \mathbf{B}^{*,+}$, and $\mathcal{M}^+ = \mathrm{M}^+ \cap \mathcal{M}$. Following Dynkin [18] we will say, for

every measure Q on $(\mathbf{M}^+, \mathcal{M}^+)$, that

$$x \in \mathbf{B}^+ \longrightarrow \int_{\mathbf{M}^+} \mathrm{e}^{-m(x)} Q(dm) \tag{2.51}$$

is the *Laplace transform of the measure* Q.

Let $(T_t;\ t \geq 0)$ be a positive, M-Pettis integrable semigroup, as defined in Definition 2.1.3, and let $L : \mathbf{B}^+ \longrightarrow \mathbf{B}$ be a SOLP-operator (see Definition 2.1.4). Then there is a unique solution $(V_t;\ t \geq 0)$ of 2.28, which is a positive, (in general nonlinear) semigroup (see Theorem 2.1.4).

Definition 2.2.1 *A family of kernels* $Q_t(m, dn)$ *on the measurable space* $(\mathbf{M}^+, \mathcal{M}^+)$ *is a* (T_t, L)-supersemigroup *if, for every* $m \in \mathbf{M}^+$ *and* $x \in \mathbf{B}^+$,

$$\int_{\mathbf{M}^+} \mathrm{e}^{-n(x)} Q_t(m, dn) = \mathrm{e}^{-m(V_t x)} \quad . \tag{2.52}$$

The existence and the uniqueness of such semigroups in the generality of Definition 2.2.1 is not known. However, in some special cases it has been studied by several authors (see [10] for detailed bibliography). The case of (T_t, L)-supersemigroup, where (T_t) is a Markov semigroup, and L is, so called, branching mechanism has been studied by Watanabe, Dawson, Dynkin, Fitzsimmons and others. We will restrict our consideration to this situation.

Let $(E, \mathcal{E})$ be a standard measurable space, i.e., there is a Polish space $(Z, \mathcal{B}(Z))$, with Borel σ-algebra, which is Borel isomorphic to $(E, \mathcal{E})$. Recall ([6], Theorem 8.3.6.) that any two Polish spaces of the same cardinality are Borel isomorphic, and that a Polish space is either countable or has the cardinality of the continuum. It follows that a measurable space is standard if and only if it is Borel isomorphic to a compact metric space with Borel σ-algebra. Therefore, we can introduce a metric on E, such that E becomes compact and $\mathcal{E}$ is the σ-algebra of Borel sets with respect

to the metric. Notice that our notion of a standard space is equivalent to the notion of Lusinian measurable space in [54] (our terminology is from [6] or [18]).

We will denote the set of (respectively, bounded, bounded nonnegative) real-valued $\mathcal{E}$-measurable functions on E by (respectively, $b\mathcal{E}$, $bp\mathcal{E}$) $\mathcal{E}$. Then $\mathbf{B} = b\mathcal{E}$ with the supremum norm, and with the partial order inherited from the real line is a Banach lattice, where $\mathbf{B}^+ = bp\mathcal{E}$. If $\mathbf{M}$ is the space of all finite signed measures on $(E, \mathcal{E})$, then $\mathbf{M}$ satisfies 2.10-2.13, and $\mathbf{M}^+$ is the set of all finite (positive) measures on $(E, \mathcal{E})$. See Example 2.1.2 for all these results. For $f \in b\mathcal{E}$ and $\mu \in \mathbf{M}$ we will denote the integral $\int_E f d\mu$ by $\langle \mu, f \rangle$.

Remark 2.2.1 There exists a metric d on $\mathbf{M}^+$ (so called Prohorov metric) such that $d(\mu_n, \mu) \to 0$ if and only if $\mu_n \xrightarrow{w} \mu$; where $\mu_n \xrightarrow{w} \mu$ means weak (Bernoulli) convergence, i.e., for every bounded and continuous f, $\langle \mu_n, f \rangle \to \langle \mu, f \rangle$.

It is important that we understand statements about $(\mathbf{M}^+, \mathcal{M}^+)$ properly. If we talk about measurability properties of $\mathbf{M}^+$ then no topology (or metric) is needed, since the definition of $\mathcal{M}^+$ is purely measure-theoretic. But, if we talk about the Prohorov metric (or weak convergence) then we assume certain topology on E. If we do not specify this topology, it means that we consider E as a compact metric space described above. Notice that in this case $\mathbf{M}^+$ with Prohorov metric is a locally compact, complete, separable metric space and its Borel σ-algebra is exactly $\mathcal{M}^+$. In particular, it shows that $(\mathbf{M}^+, \mathcal{M}^+)$ is a standard measurable space. For the proofs of results mentioned above see, for example, [7], pp.607-638.

However, sometimes we can consider E with its "natural" topology (if it exists, of course), which is not necessarily compact. In such a case, we have to deal with topological properties of $\mathbf{M}^+$ separately.□

Let $\xi = (\xi_t;\ t \geq 0)$ be a time-homogeneous Markov process, with state space $(E, \mathcal{E})$, transition function $p_t(x, dy)$, and corresponding semigroup $(T_t;\ t \geq 0)$. We assume that, for every $A \in \mathcal{E}$, $(t, x) \mapsto p_t(x, A)$ is measurable. (T_t) is a contraction semigroup, and, as we have seen in Example 2.1.2, (T_t) is a positive, M-Pettis integrable semigroup.

Let $b \in b\mathcal{E}$, $c \in bp\mathcal{E}$, and $n(x, du)$ be a positive kernel from $(E, \mathcal{E})$ to the measurable space of positive real numbers $((0, +\infty), \mathcal{B}((0, +\infty)))$ such that

$$x \longrightarrow \int_0^{+\infty} u\, n(x, du) \in bp\mathcal{E} \quad . \tag{2.53}$$

In particular, 2.53 shows that there is a constant $k > 0$ such that, for every $x \in E$ and $\varepsilon > 0$,

$$n(x, [\varepsilon, +\infty)) \leq \frac{k}{\varepsilon} \quad . \tag{2.54}$$

We define a *branching mechanism* as a function $\Psi : E \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ of the form

$$\Psi(x, \lambda) = -b(x)\lambda - c(x)\lambda^2 + \int_0^{+\infty} (1 - \mathrm{e}^{-\lambda u}) n(x, du) \quad . \tag{2.55}$$

For $f \in bp\mathcal{E}$ we define $\Psi f \in \mathcal{E}$ by

$$(\Psi f)(x) = \Psi(x, f(x)) \quad . \tag{2.56}$$

Using 2.53 and the fact that, for $z \geq 0$, $1 - \mathrm{e}^{-z} \leq z$, we obtain that, for every $f \in bp\mathcal{E}$,

$$\|\Psi f\| \leq \|f\| \cdot \left(\|b\| + \|c\| \cdot \|f\| + \left\| \int_0^{+\infty} u\, n(x, du) \right\| \right) \quad . \tag{2.57}$$

Notice that the factor $\|c\| \cdot \|f\|$ prevents us from concluding that Ψ is bounded, but 2.57 does show that $\Psi : bp\mathcal{E} \to b\mathcal{E}$, and $\Psi 0 = 0$. Notice also that directly from the definition 2.55 follows that Ψ is M-measurable. Let us prove that Ψ is locally Lipschitz. Applying the mean value theorem on $z \mapsto \exp(-z)$, we obtain

$$|\,(\Psi f)(x) - (\Psi g)(x)\,| \leq |\,f(x) - g(x)\,| \cdot$$

$$\cdot \left(|b(x)| + c(x)|\, f(x) + g(x)\,| + \int_0^{+\infty} u\, n(x, du) \right) \quad ,$$

which, together with 2.53, shows that Ψ is locally Lipschitz.

It is possible to rewrite a branching mechanism in the form

$$\Psi(x, \lambda) = -\bar{b}(x)\lambda - c(x)\lambda^2 + \int_0^{+\infty} (1 - \mathrm{e}^{-\lambda u} - \lambda u) n(x, du) \quad , \tag{2.58}$$

where

$$\bar{b}(x) = b(x) - \int_0^{+\infty} u\, n(x, du) \quad . \tag{2.59}$$

Notice that the assumption 2.53 was crucial here, and that it shows that $\bar{b} \in b\mathcal{E}$. Since $1 - \mathrm{e}^{-\lambda u} - \lambda u \leq 0$, for $\lambda \geq 0$, it follows that, for every $f \in bp\mathcal{E}$,

$$(\Psi f)^+ = (\bar{b})^- \cdot f \quad . \tag{2.60}$$

This, together with our choice of Banach lattice $b\mathcal{E}$, shows that an operator Ψ satisfies condition b) from Theorem 2.1.2, and that $\|\Psi^+\| = \|\bar{b}^-\|$.

Let $K > 0$ be a positive number. The fact that the derivative

$$\left| \frac{d}{d\lambda} \Psi(x, \lambda) \right| \leq \|b\| + 2\lambda \|c\| + \left\| \int_0^{+\infty} u n(x, du) \right\|$$

implies that

$$a = a(K) = \sup_{x \in E, \lambda \in [0, K]} \left\{ \left| \frac{d}{d\lambda} \Psi(x, \lambda) \right| \right\}$$

is finite and nonnegative. Then, for every $x \in E$, $\lambda \mapsto \Psi(x, \lambda) + a\lambda$ is nonnegative and increasing on $[0, K]$, which implies that Ψ satisfies condition 2.44. Hence, we proved that Ψ is a SOLP-operator. It follows now, from Theorem 2.1.4, that there exists one and only one solution $V_t : bp\mathcal{E} \to bp\mathcal{E}$ of an integral equation

$$(V_t f)(x) = (T_t f)(x) + \int_0^t T_s[\Psi V_{t-s} f](x)\, ds \quad , \tag{2.61}$$

and that $(V_t;\ t \geq 0)$ is a semigroup, $(t, x) \mapsto (V_t f)(x)$ is measurable, and, for every $f \in bp\mathcal{E}$,

$$\|V_t f\| \leq \mathrm{e}^{\|\bar{b}^-\| \cdot t} \|f\| \quad . \tag{2.62}$$

This result was obtained by Fitzsimmons, [28], using similar method as our iteration method in section 2.1., but applied only to the case of $bp\mathcal{E}$ and Ψ, and by Dynkin, [18], in the case of $\Psi(x,\lambda) = -\lambda^2$, and by different method. Here we just proved that their results are special cases of our general result given in section 2.1. However, the semigroups of this type were studied in connection with measure-valued processes for the first time in Watanabe, [56], and in several works by Dawson (see [10] for complete bibliography). It must be mentioned that the basic ideas for such development were already given in Jiřina, [36]. Following Dynkin's terminology we will give a special name to a (T_t, Ψ)-supersemigroup.

Definition 2.2.2 *A (T_t, Ψ)-supersemigroup, where Ψ is a branching mechanism, and (T_t) is a semigroup which corresponds to a Markov process (ξ_t) is called a* Dawson-Watanabe supersemigroup.
Markov process $(X_t;\ t \geq 0)$ with the state space $(\mathrm{M}^+, \mathcal{M}^+)$ is called a Dawson-Watanabe superprocess *over (ξ_t), with branching mechanism Ψ, if its semigroup is a Dawson-Watanabe supersemigroup.*

We will use the same name (Dawson-Watanabe supersemigroup) to indicate the transition function $Q_t(\mu, d\nu)$ on $(\mathrm{M}^+, \mathcal{M}^+)$, and the actual semigroup of operators

$$(Q_t F)(\mu) = \int_{\mathrm{M}^+} F(\nu) Q_t(\mu, d\nu) \quad . \tag{2.63}$$

Also, when it is clear what Ψ and (ξ_t) are, we will refer to (X_t) as to a superprocess only.

The question of the existence and uniqueness of the superprocess comes immediately into consideration. Notice that once we have a supersemigroup $Q_t(\mu, d\nu)$, then, since $(\mathrm{M}^+, \mathcal{M}^+)$ is a standard space, Kolmogorov existence theorem guarantees the existence of the superprocess (X_t). Of course, for a moment we do not say anything about regularity properties of (X_t). We will come to this in the next section.

Therefore, the question is the existence and uniqueness of $Q_t(\mu, d\nu)$, given by 2.52. Notice that the Markov property of $Q_t(\mu, d\nu)$ is the consequence of the semigroup property of (V_t). Notice also that $\mu \mapsto \exp(-\langle \mu, V_t f \rangle)$ is $\mathcal{M}^+$-measurable. An application of the monotone class theorem for the multiplicative systems of functions shows that $\mu \mapsto Q_t(\mu, d\nu)$ is $\mathcal{M}^+$-measurable , i.e., $Q_t(\mu, d\nu)$ must be a kernel, if it exists. $V_t 0 = 0$ shows that every $Q_t(\mu, d\nu)$ is a probability measure. That the Laplace transform determines a measure uniquely (on a standard space) has been proved in [18], pp.264-265. Therefore, it is enough to prove that for any μ and $t \geq 0$ there exists a measure $Q_t(\mu, d\nu)$ which satisfies 2.52. The solution to this problem has been given in [28]. However, some parts of the construction from [28] are more demanding than it may look at first sight. This and the wish for completeness of the work led us to include these proofs here.

Notice that $(\mathbf{B}^+, +)$ forms a two-divisible semigroup, i.e., for every $x \in \mathbf{B}^+$, there exists $y \in \mathbf{B}^+$, such that $x = y + y$ (of course, $y = (1/2)x$). Hence, the Laplace transform is a positive definite function on the semigroup $\mathbf{B}^+$ (see 2.51). The theory of positive definite functions on a two-divisible semigroup $(S, +)$ is developed in [2]. It has been proved there that for every bounded positive definite function $\varphi : S \longrightarrow \mathbb{R}$ there exists a unique positive Radon measure γ on $\hat{S}$, such that

$$\varphi(s) = \int_{\hat{S}} \varrho(s)\gamma(d\varrho) \quad , \tag{2.64}$$

where $\hat{S}$ is the set of bounded semicharacters, i.e., of the functions $\varrho : S \longrightarrow [0, 1]$ such that $\varrho(0) = 1$, and $\varrho(s_1 + s_2) = \varrho(s_1) \cdot \varrho(s_2)$. Moreover, if $\varphi(0) = 1$, then γ is a probability measure. Recall also ([2], Theorem 3.2.2.) that positive definite functions are negative exponentials of negative definite functions, i.e., of the functions $\theta : S \longrightarrow \mathbb{R}$, such that

$$\sum_{i,j=1}^{n} a_i a_j \theta(s_i + s_j) \leq 0 \quad , \tag{2.65}$$

whenever $\{a_1, \ldots, a_n\} \subseteq \mathbb{R}$ sum to zero, $\{s_1, \ldots, s_n\} \subseteq S$, and $n \geq 2$. Therefore, in application of these theories on the solution (V_t) of 2.28, one has to prove first that $x \mapsto m[V_t x]$ is negative definite. In terms of branching mechanism and in the notations of $bp\mathcal{E}$, one has to prove that $f \mapsto (V_t f)(x)$ is negative definite, for every $x \in E$ (do not confuse $x \in E$ and $x \in \mathbf{B}$!).

Remark 2.2.2 The statement about negative definitness of $f \mapsto (V_t f)(x)$ is given in [28], p.341, and it is suggested there that the proof is standard. Through private communication with the author of [28] we pointed out some possible difficulties in the proof. After some discussion both sides offered correct proof (which happened to be the same on both sides). This is still a standard proof, but different from the first proof that was suggested. We present the proof below.□

Proposition 2.2.1 *Let* $\mathbf{B}^+ = bp\mathcal{E}$, *and* $L = \Psi$. *If* $(V_t;\ t \geq 0)$ *is a solution of 2.28, then, for every* $t \geq 0$ *and* $x \in E$,

$$f \mapsto (V_t f)(x) \tag{2.66}$$

is negative definite on $(bp\mathcal{E}, +)$.

Proof. Consider, for every $m \in \mathbb{N}$, the branching mechanism

$$\Psi_m(x, \lambda) = -\bar{b}(x)\lambda + \int_0^{+\infty} (1 - \mathrm{e}^{-\lambda u} - \lambda u)\tilde{n}(x, du) \quad , \tag{2.67}$$

where

$$\tilde{n}(x, du) = n(x, du) + 2m^2 c(x)\delta_{\{\frac{1}{m}\}} \quad , \tag{2.68}$$

where $\delta_{\{a\}}$ is the point-mass at a. Notice that, for every x and λ, $\Psi_m(x, \lambda) \to \Psi(x, \lambda)$, when $m \to +\infty$. Hence, for every f and x, there is a subsequence $\{m_k\}$ such that $(W_t^{m_k} f)(x) \to (V_t f)(x)$, where $(W_t^m;\ t \geq 0)$ is the solution of 2.28 for $L = \Psi_m$.

Therefore, it is enough to prove negative definiteness of the solution of 2.28 where L is the branching mechanism of the form 2.67. Notice that, because of 2.53, such branching mechanism can be written in the form

$$(x,\lambda) \mapsto -\hat{b}(x)\lambda + \int_0^{+\infty} (1 - e^{-\lambda u})\hat{n}(x, du) \quad . \tag{2.69}$$

For a fixed $x \in E$, the function above is a Bernstein function (see [3]). The composition of a Bernstein function with a negative definite function is again negative definite (see [3], and [2] p.114). If $(W_t;\ t \geq 0)$ is the solution of 2.28 with the branching mechanism of the form 2.69, then $(W_t f)(x) = \lim_{l\to\infty}(W_t^l f)(x)$, where W_t^l are defined inductively by 2.45. Since $W_t^0 \equiv 0$ is negative definite, and linear function $(T_t f)(x)$ is negative definite, we prove inductively that $(W_t^l f)(x)$ is negative definite, for every l (applying the above fact about Bernstein function).

$Q.E.D.$

Remark 2.2.3 There is a possibility for an easy mistake in the proof above. In some cases a function, say θ, can be regarded as a function on a group and on a semigroup, as well. Typical example is the function $\theta(x) = -x^2$. θ is the negative definite function in the sense of a semigroup $(\mathbb{R}_+, +)$, but θ is not the negative definite function in the sense of a group $(\mathbb{R}, +)$. However, when we compose θ the situation changes. θ composed with the negative definite function in the sense of a semigroup $(\mathbb{R}_+, +)$ is *not* necessarily a negative definite function on $(\mathbb{R}_+, +)$. For example, $x \mapsto \ln(1 + x)$ is a negative definite function on $(\mathbb{R}_+, +)$ (to prove that consider the Laplace transform of the Γ-distribution), but $x \mapsto -[\ln(1 + x)]^2$ is not.□

The last part of the construction of $Q_t(\mu, d\nu)$ is given in [28]. For the sake of completeness we will present it here (in some nonessential details our argument will be different).

Notice that, for every $x \in E$, $\lim_{n\to\infty}(V_t \frac{1}{n})(x) = 0$, by 2.62. By Proposition 2.2.1 and 2.64 there is a probability measure γ on $\hat{S}$ (where $S = bp\mathcal{E}$), such that

$$e^{-\langle \mu, V_t f\rangle} = \int_{\hat{S}} \varrho(f)\gamma(d\varrho) \quad , \tag{2.70}$$

and γ is unique. Since $\varrho(1/n) = \varrho(1)^{1/n}$, it follows that γ is concentrated on

$$\hat{S}_+ = \{\, \varrho \in \hat{S} \mid \varrho(s) > 0, \text{ for every } s \in S \,\} \quad .$$

If $f_n \searrow 0$, $f_n \in bp\mathcal{E}$, then $(V_t f_n)(x) \searrow g(t) \geq 0$, since V_t is monotone, by 2.70 (in the sense that if $f, g \in bp\mathcal{E}$, $f \leq g$, then $V_t f \leq V_t g$). Using 2.28 we get

$$g(t) = \int_0^t T_s[\Psi g(t-s)]ds \quad .$$

By the uniqueness of the solution of 2.28, we obtain $g(t) \equiv 0$, i.e., $V_t f_n \searrow 0$. By 2.70, $\varrho(f_n) \nearrow 1$ γ-(a.e.).

If we fix $f \in bp\mathcal{E}$, $\varrho \in \hat{S}_+$, and, for every $t \geq 0$, we denote $\varrho(tf)$ by $g(t)$, then $g(0) = 1$, $g : [0, +\infty) \longrightarrow (0, 1]$, and $g(t+s) = g(t)g(s)$. Hence, $g(t) = \exp(-at)$, where $a = -\ln \varrho(f) > 0$.

Now, let us define $T : bp\mathcal{E} \longrightarrow b\mathcal{B}(\hat{S}_+)$ by

$$(Tf)(\varrho) = -\ln \varrho(f) \quad , \tag{2.71}$$

where $\mathcal{B}(\hat{S}_+)$ is the σ-algebra of Borel sets on $\hat{S}_+$. We just proved that T is additive, homogeneous for positive real numbers, and, if $f_n \searrow 0$, then $Tf_n \searrow 0$ γ-(a.e.). Very simple adjustment of the theorem in [31] for the case of standard spaces, shows that there exists a kernel K on $\hat{S}_+ \times \mathcal{E}$, bounded and positive, such that, for every $f \in bp\mathcal{E}$,

$$Kf = Tf \quad \gamma-(\text{a.e.}) \, . \tag{2.72}$$

Hence, for every $\varrho \in \hat{S}_+$ there is a measure K_ϱ in M^+, and, by definition of $\mathcal{M}^+$, $\varrho \mapsto K_\varrho$ is $\mathcal{B}(\hat{S}_+)/\mathcal{M}^+$-measurable. Let Q be the image measure of γ under $\varrho \mapsto K_\varrho$.

Then, by 2.70 and 2.72,

$$e^{-\langle \mu, V_t f\rangle} = \int_{M_+} e^{-\langle \nu, f\rangle} Q(d\nu) \quad , \tag{2.73}$$

and the construction of a Markov transition function $Q_t(\mu, d\nu)$, which is also a supersemigroup, is complete. In other words we proved the following theorem:

Theorem 2.2.1 (Dynkin, Fitzsimmons) Let $(E, \mathcal{E})$ be a standard space. For every time-homogeneous Markov process ξ on E, such that $(t, x) \mapsto p_t(x, dy)$ is measurable, and, for every branching mechanism Ψ, which satisfies 2.53, there exists one and only one supersemigroup $Q_t(\mu, d\nu)$.

Remark 2.2.4 Locally Lipschitz property was crucial in the construction of the superprocesses. We were using 2.53 to obtain it. However, it is worth mentioning that locally Lipschitz property of Ψ follows under weaker condition than 2.53, i.e., under the condition

$$x \mapsto \int_0^1 u^2 n(x, du) + \int_1^\infty u n(x, du) \in bp\mathcal{E} \quad .$$

But in this case we can not transform 2.55 into 2.58 and vice versa. Therefore, it leads to different constructions where our general approach, developed in section 2.1., cannot be applied so easily. For the construction of superprocesses under this weaker condition we refer to [10], Chapter 3.

Let us also mention that in [28] the author assumes 2.53 and one more condition, i.e., that

$$x \mapsto \int_0^\infty u^2 n(x, du) \in bp\mathcal{E} \quad ,$$

but he uses it only for the regularity properties of the superprocesses. For the construction of the supersemigroup it is not needed.□

Let us mention that there is a nice (and important, as we will see later), probabilistic interpretation of the branching mechanism Ψ. Notice that, for every $x \in E$,

$\lambda \mapsto \Psi(x,\lambda)$ is a Log-Laplace function of an infinitely divisible distribution (see [27], XIII.7.). More precisely,

$$e^{-\Psi(x,\lambda)} = E\left(e^{-\lambda(D_x+N_x+P_x)}\right) \quad , \tag{2.74}$$

where D_x, N_x, P_x are independent random variables with values in $\mathbb{R}$, such that D_x is a point-mass at $-b(x)$, N_x is a centered normal distribution with $\mathrm{Var}(N_x) = 2c(x)$, and P_x is a positive infinitely divisible distribution with finite expectation

$$E(P_x) = \int_0^\infty u n(x,du) \quad . \tag{2.75}$$

Recall also (see [35] for example), that there is a unique temporally homogeneous Lévy process, i.e., a process with stationary independent increments with no point of fixed discontinuity and whose sample paths are càdlàg, say $(L_t^{\Psi(x,\cdot)})$, such that

$$e^{-\Psi(x,\lambda)} = E\left(e^{-\lambda L_1^{\Psi(x,\cdot)}}\right) \quad . \tag{2.76}$$

If $\Psi(x,\lambda) = -\lambda$ then $(L_t^{\Psi(x,\cdot)})$ is a uniform motion to the left. If $\Psi(x,\lambda) = -\lambda^2$ then $(L_t^{\Psi(x,\cdot)})$ is a standard Brownian motion. If $\Psi(x,\lambda) = 1 - e^{-\lambda}$ then $(L_t^{\Psi(x,\cdot)})$ is a Poisson process.

We will apply Lemma 2.1.3 on superprocesses to obtain an interesting result, which suggests that in some cases we can study always the same (and extremaly simple) Markov process, and change the perturbing operator accordingly, to get Dawson-Watanabe superprocess.

Let us denote the trivial semigroup, which consists only of identity operators, by $(\,Id_t;\, t \geq 0\,)$. This semigroup is defined on $b\mathcal{E}$, but also it is defined on any subspace of $b\mathcal{E}$. To this semigroup corresponds a trivial Markov process on $(E,\mathcal{E})$, which "stays forever at the point". Let us denote this Markov process by $(\,d_t;\, t \geq 0\,)$. Then, for every $t \geq 0$, $d_t \equiv d_0$.

Theorem 2.2.2 *Let ξ be a Markov process, such that there exists an* M*-invariant, bounded, linear operator A, such that, for every $t \geq 0$, $T_t = \exp(tA)$. Then there exists a Dawson-Watanabe supersemigroup over ξ, with branching mechanism Ψ, and is equal to a $(Id_t, \Psi + A)$-supersemigroup.*

Proof. By our assumptions, for every $G \in \mathcal{E}$,

$$p_t(x, G) = \mathrm{P}^x(\xi_t \in G) = (T_t 1_G)(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!}(A^n 1_G)(x) \ ,$$

which shows that $(t, x) \mapsto p_t(x, dy)$ is measurable. By Theorem 2.2.1, there exists one and only one supersemigroup $Q_t(\mu, d\nu)$, such that its Laplace transform is $\exp(-\langle \mu, V_t f \rangle)$, where, for every $f \in bp\mathcal{E}$,

$$V_t f = T_t f + (\mathrm{MP}) \int_{(0,t)} T_s[\Psi(V_{t-s} f)] ds \quad .$$

As we have seen in Example 2.1.2 (T_t) is a positive, M-Pettis integrable semigroup, and

$$T_t f = Id_t f + (\mathrm{MP}) \int_{(0,t)} T_s[A(Id_{t-s} f)] ds \quad ,$$

which implies, since A and T_s commute,

$$Id_t f = T_t f + (\mathrm{MP}) \int_{(0,t)} Id_s[(-A)(T_{t-s} f)] ds \quad .$$

Since A is M-invariant, we are in the situation of Lemma 2.1.3, which implies that

$$V_t f = Id_t f + (\mathrm{MP}) \int_{(0,t)} Id_s[(\Psi - (-A))(V_{t-s} f)] ds =$$

$$= f + (\mathrm{MP}) \int_{(0,t)} (\Psi + A)(V_s f) ds \quad . \tag{2.77}$$

Q.E.D.

An example of a Markov process which satisfies Theorem 2.2.2 is a Poisson process. Another interesting class of processes which satisfy Theorem 2.2.2 consists of Markov

chains with differentiable transition probabilities. We will describe this case more carefully here.

Let $E = \{1, 2, \ldots, n\}$ and $\mathcal{E} = 2^E$. In this case $\mathbf{B} = b\mathcal{E}$ is equal to $\mathbb{R}^n$, and $\mathbf{B}^+ = bp\mathcal{E}$ is equal to

$$\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, \text{ for every } i\} \quad .$$

However $\mathbf{M} = \mathbf{B}^*$ is also equal to $\mathbb{R}^n$, and $\mathbf{M}^+ = \mathbb{R}^n_+$. Notice that $\mathcal{M} = \mathcal{B}(\mathbb{R}^n)$ and $\mathcal{M}^+ = \mathcal{B}(\mathbb{R}^n_+)$. To distinguish between measures and functions we will use notation $f, g, \ldots, f = (f_1, \ldots, f_n)$ for functions, and $\mu, \nu, \ldots, \mu = (\mu_1, \ldots, \mu_n)$ for measures. Of course $\mu_i = \mu(\{i\})$, and $f_i = f(i)$, and $\langle \mu, f \rangle = \sum_{i=1}^n \mu_i f_i$. In this case T_t is a matrix with elements $p_{ij}(t)$, $i, j \in E$, and the measurability condition, on $(t, x) \mapsto p_t(x, dy)$, is equivalent to the condition that, for every i, j, $t \mapsto p_{ij}(t)$ is measurable. By Theorem 2.2.1, the Dawson-Watanabe supersemigroup exists already in this case, but we would like to consider the cases in which requirements for Theorem 2.2.2 are fulfilled. Since we are in a finite dimensional case, these requirements are equivalent to the condition that (T_t) is a Feller semigroup, which is equivalent to the condition that, for every i, j,

$$\lim_{t \to 0} p_{ij}(t) = \delta_{ij} \quad . \tag{2.78}$$

In the following we will restrict our attention to such semigroups (T_t). In such a case there exists a matrix A, an infinitesimal generator, such that $A = (a_{ij})_1^n$ and $T_t = \exp(tA)$, and

$$a_{ij} \geq 0, \text{ for } i \neq j, \ \sum_{j=1}^n a_{ij} = 0 \quad . \tag{2.79}$$

Compare this condition to the end of section 2.1.

A good reference for these results is [5]. In [5] a transition matrix (T_t), which satisfies our conditions, is called a *standard transition matrix*, and the matrix A is

called a Q-matrix. We will refer to (T_t) as a standard transition matrix, but to A as an infinitesimal generator (we use Q for supersemigroups).

Typical example of a Markov chain which has a supersemigroup, but does not have a standard transition matrix (and, therefore, does not satisfy Theorem 2.2.2) is, for $n = 2$,

$$T_t = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \quad ; \; t > 0 \; . \tag{2.80}$$

2.3 Kac Semigroup and Linear Case

In this section we will describe Dawson-Watanabe superprocesses over very general class of Markov processes with linear branching mechanism. In this case Dawson-Watanabe supersemigroups are related to a familiar notion of Kac semigroup, and they can be used as upper and lower bounds (in some sense) for the general case of branching mechanism.

Let (ξ_t) be a Borel right Markov process on a Lusin state space $(E, \mathcal{E})$ with transition probability $p_t(x, dy)$ (see [54]). In this case $\mathcal{E}$ is the σ-algebra of Borel sets, (ξ_t) is a.s. right continuous, and $(t, x) \mapsto p_t(x, dy)$ is measurable. We will also assume that, for the corresponding Markov semigroup (T_t), $T_t 1 \equiv 1$. Let $\bar{k} \in b\mathcal{E}$ be a bounded measurable function (not necessarily nonnegative). We define, for every $G \in \mathcal{E}$,

$$k_t(x, G) = \mathrm{E}^x \left[1_G(\xi_t) \cdot \mathrm{e}^{-\int_0^t \bar{k}(\xi_u) du} \right] \quad . \tag{2.81}$$

Then, by monotone class arguments, $k_t(x, dy)$ is a positive kernel, such that $(t, x) \mapsto k_t(x, dy)$ is measurable. This kernel defines a semigroup $(K_t; \, t \geq 0)$ given, for every $f \in bp\mathcal{E}$, by

$$(K_t f)(x) = \int_E f(y) k_t(x, dy) = \mathrm{E}^x \left[f(\xi_t) \cdot \mathrm{e}^{-\int_0^t \bar{k}(\xi_u) du} \right] \quad . \tag{2.82}$$

Notice that (K_t) is not necessarily Markovian, since $\bar{k}$ can be negative, but (K_t) is bounded by $\exp(\|\bar{k}\| \cdot t)$ and is positive. From Example 2.1.2 we obtain that (T_t) and

(K_t) are both positive, M-Pettis integrable semigroups (in the sense of Definition 2.1.3), where M is the set of all finite signed measures on $(E, \mathcal{E})$.

Recall (see, for example, [49], Chapter 3) that, in the case of Brownian motion and $\bar{k} \geq 0$, the semigroup (K_t) satisfies, so called, Kac (or Feynman-Kac) formula. In this work we will refer to the semigroup (K_t), defined by 2.82, as a *Kac semigroup*.

Consider now a linear operator $B : b\mathcal{E} \longrightarrow b\mathcal{E}$, defined by

$$(Bf)(x) = -\bar{k}(x) \cdot f(x) \quad . \tag{2.83}$$

It is obvious that B is an M-invariant operator and an SOLP-operator (see Definition 2.1.4). We claim that (K_t) and (T_t) satisfy the conditions of Lemma 2.1.3. It is enough to prove the following lemma:

Lemma 2.3.1 *For every $f \in b\mathcal{E}$ and $x \in E$,*

$$(K_t f)(x) = (T_t f)(x) + \int_0^t K_s[B(T_{t-s} f)](x) ds \quad . \tag{2.84}$$

Proof. Using 2.82 and Markov property, we obtain

$$(K_t f)(x) = \mathrm{E}^x \left[f(\xi_t) \cdot \mathrm{e}^{-\int_0^t \bar{k}(\xi_u) du} \right] =$$

$$= \mathrm{E}^x \left[f(\xi_t) \cdot \left(1 - \int_0^t \bar{k}(\xi_u) \cdot \mathrm{e}^{-\int_0^u \bar{k}(\xi_s) ds} \, du \right) \right] =$$

$$= (T_t f)(x) + \int_0^t \mathrm{E}^x \left[-\bar{k}(\xi_u) \cdot f(\xi_t) \cdot \mathrm{e}^{-\int_0^u \bar{k}(\xi_s) ds} \right] du =$$

$$= (T_t f)(x) + \int_0^t \mathrm{E}^x \left[-\bar{k}(\xi_u) \cdot f(\xi_t) \cdot \mathrm{e}^{-\int_0^u \bar{k}(\xi_s) ds} / \mathcal{F}_u \right] du =$$

$$= (T_t f)(x) + \int_0^t \mathrm{E}^x \left[-\bar{k}(\xi_u) \cdot \mathrm{E}^{\xi_u} [f(\xi_{t-u})] \cdot \mathrm{e}^{-\int_0^u \bar{k}(\xi_s) ds} \right] du =$$

$$= (T_t f)(x) + \int_0^t K_u[-\bar{k} \cdot (T_{t-u} f)](x) du \quad .$$

Q.E.D.

Consider now a branching mechanism $\Psi_1(x,\lambda) = -b(x)\lambda$, where $b \in b\mathcal{E}$. Let us denote the corresponding solution of 2.61 by $(W_t;\, t \geq 0)$. Let $(K_t;\, t \geq 0)$ be the Kac semigroup, where $\bar{k} = b$. By Lemma 2.3.1, we can apply Lemma 2.1.3 on these semigroups. Since $\Psi_1 - (-bf) = 0$, we get $W_t = K_t$, for every $t \geq 0$, i.e., Kac semigroup is a solution of 2.61.

Remark 2.3.1 One can think that we have proved already in Lemma 2.3.1 that (K_t) is a solution for 2.61. But it is not so. Compare 2.61 (i.e., 2.28) with 2.84 (i.e., 2.41), and notice that semigroups appear in different "order" under integral. In many cases it will be the same solution, but one has to prove it.

However, the solution of 2.61 for $\Psi_1(x,\lambda) = -b(x)\lambda$ can be computed directly, and this is a known result (see [28]). We just wanted to emphasize that it is also a consequence of our general method developed in section 2.1.□

Once we know that (K_t) is a solution of 2.61, it is easy to construct the corresponding superprocess. We will introduce some notation first. For a measure $\mu \in \mathrm{M}^+$, we denote by Δ_μ a point mass at μ. Of course, Δ_μ is a probability measure on $(\mathrm{M}^+, \mathcal{M}^+)$. As usual, the point mass at $x \in E$ we will denote by δ_x. So, δ_x is a measure on $(E,\mathcal{E})$, i.e., $\delta_x \in \mathrm{M}^+$, and it make sense to consider Δ_{δ_x}. For $\mu \in \mathrm{M}^+$ we denote by μK_t a measure on $(E,\mathcal{E})$, defined by,

$$\langle \mu K_t, f\rangle = \langle \mu, K_t f\rangle = \int_E \mathrm{E}^x \left[f(\xi_t)\, \mathrm{e}^{-\int_0^t b(\xi_u)du} \right] \mu(dx)\,. \tag{2.85}$$

Obviously, $\mu K_t \in \mathrm{M}^+$. Formula 2.85 immediately shows that we can describe the linear case, $\Psi_1(x,\lambda) = -b(x)\lambda$, completely, i.e., that we proved the following theorem.

Theorem 2.3.1 *(linear case) A Dawson-Watanabe superprocess, over (ξ_t) with branching mechanism $\Psi_1(x,\lambda) = -b(x)\lambda$, is a deterministic Markov process (X_t), defined*

by $X_t = X_0 K_t$, where (K_t) is a Kac semigroup (with $\bar{k} = b$). The supersemigroup is given by the formula

$$Q_t(\mu, d\nu) = \Delta_{\mu K_t}(d\nu) \quad . \tag{2.86}$$

Consider now a general form of branching mechanism $\Psi(x, \lambda)$, given by 2.55. It can be rewritten in the form 2.58. Let $\Psi_2(x, \lambda) = \bar{b}^-(x)\lambda$ be negative of the positive part of Ψ. Let (K_t) be the Kac semigroup, where $\bar{k} = -\bar{b}^-$. Let (V_t) be the solution of 2.61 for Ψ, and $Q_t(\mu, d\nu)$ the corresponding supersemigroup. It has been proved in [28], p.342, that the first moment of $Q_t(\mu, d\nu)$ is given by

$$\int_{M^+} \langle \nu, f \rangle Q_t(\mu, d\nu) = \langle \mu, \mathrm{E}^x \left[f(\xi_t)\, \mathrm{e}^{-\int_0^t \bar{b}(\xi_s) ds} \right] \rangle \ . \tag{2.87}$$

These facts lead to one more application of Lemma 2.1.3, which gives results stronger than 2.62 on bounds of (V_t).

<u>*Theorem 2.3.2*</u> *For every $t \geq 0$, $x \in E$, and $f \in bp\mathcal{E}$, the following properties are satisfied:*

$$0 \leq (V_t f)(x) \leq (K_t f)(x) = \mathrm{E}^x \left[f(\xi_t)\, \mathrm{e}^{\int_0^t \bar{b}^-(\xi_s) ds} \right] \ . \tag{2.88}$$

$$(V_t f)(x) = 0 \iff (T_t f)(x) = 0 \iff (K_t f)(x) = 0 \quad . \tag{2.89}$$

In particular, if $\bar{b}^- \equiv 0$, then $0 \leq (V_t f)(x) \leq (T_t f)(x)$.

<u>*Proof.*</u> Consider $\Psi - \Psi_2$. It is a branching mechanism again, but with the property that, for every $x \in E$ and $\lambda \geq 0$, $(\Psi - \Psi_2)(x, \lambda) \leq 0$. By Lemma 2.3.1 (T_t) and (K_t) satisfy the conditions of Lemma 2.1.3, which implies that

$$(V_t f)(x) = (K_t f)(x) - \int_0^t K_s[-(\Psi - \Psi_2)(V_{t-s} f)](x)\, ds \quad . \tag{2.90}$$

Since K_s, $-(\Psi - \Psi_2)$, and V_{t-s} are all positive, it follows that $(V_t f)(x) \leq (K_t f)(x)$, which proves 2.88.

To prove 2.89 we should notice first that $(V_t f)(x) \leq (K_t f)(x) \leq (T_t f)(x) \cdot \exp(\|\bar{b}^-\| \cdot t)$, which proves that

$$(T_t f)(x) = 0 \Longrightarrow (K_t f)(x) = 0 \Longrightarrow (V_t f)(x) = 0 \quad .$$

We have to prove only that $((V_t f)(x) = 0 \Longrightarrow (T_t f)(x) = 0)$. If $(V_t f)(x) = 0$, then, by 2.52, $\langle \nu, f \rangle = 0$, $Q_t(\delta_x, d\nu) - (a.e.)$ (recall that $\langle \delta_x, V_t f \rangle = (V_t f)(x)$). It follows that the first moment of $Q_t(\delta_x, d\nu)$ is zero. By 2.87, we obtain that

$$\mathrm{E}^x \left[f(\xi_t)\, \mathrm{e}^{-\int_0^t \bar{b}(\xi_s) ds} \right] = 0 \quad .$$

Since $\bar{b}$ is bounded, the exponential factor is strictly positive, and

$$f(\xi_t)\, \mathrm{e}^{-\int_0^t \bar{b}(\xi_s) ds} = 0, \ \mathrm{P}^x - (a.e.) \, .$$

It follows that $f(\xi_t) = 0, \ \mathrm{P}^x - (a.e.)$, i.e.,

$$0 = \mathrm{E}^x \left[f(\xi_t) \right] = (T_t f)(x) \quad .$$

Q.E.D.

Remark 2.3.2 It is a good place to comment on additional requirements on superprocesses assumed by some authors. In some of his papers E.B. Dynkin (see for example [18]) require, what seems to be a "natural" probabilistic condition, that, for every $B \in \mathcal{E}$,

$$\text{Probability}\{\xi_t \in B\} = \text{Expectation}[X_t(B)] \quad .$$

Recall that (X_t) is a measure-valued, so above formula makes sense. By monotone class argument, it is equivalent to

$$\langle \mu, T_t f \rangle = \int_{\mathrm{M}^+} \langle \nu, f \rangle Q_t(\mu, d\nu) \quad .$$

However, formula 2.87 shows that it is the case if and only if $\bar{b} \equiv 0$, but not always. In particular, it is true for $\Psi(x, \lambda) = -\lambda^2$, which is the case often studied by Dynkin. Despite these facts we will continue to study the general form of Ψ.□

2.4 Feller Condition and Infinitesimal Generator

Regularity properties of superprocesses have been studied by many authors. In the case of a compact metric space E, and a branching mechanism Ψ with constant coefficients, the problem has been treated already by Watanabe [56]. His approach deals with Feller properties of semigroups involved. Since $\mathbf{M}^+$ is, in this case, a locally compact space, we can deal with Feller property in a nice way. However, very often we deal with Markov processes that are defined on a locally compact Hausdorff space with a countable base. In such a case, $\mathbf{M}^+$ is not necessarily locally compact. Several authors offered ways to resolve this situation (see [9], [34], and [52], for example). Usually authors adjust the conditions according to their special needs.

In the following text it is general enough, for us, to treat the case when (ξ_t) satisfies Feller property. We believe that the most "natural" way is to extend our process onto the one-point compactification of the state space E, to develop a superprocess over it which has Feller property, and to show that such superprocess actually "lives" on the set of measures on E. Of course, the Feller property of the superprocess on the one-point compactification is the result of Watanabe. But, it follows also from our general method, developed in section 2.1., as a special case.

Let us mention that the problem of regularity properties has been treated in a very general form in [28]. It is shown there that the superprocess is a right Borel Markov process, whenever (ξ_t) is a right Borel Markov process with Lusin state space $(E, \mathcal{E})$, and Ψ satisfies condition on the second moment (see Remark 2.2.4). The temporally non-homogeneous case was treated in [17], and generalized even more in [21]. We do not need so general a Markov process (ξ_t), and, on the other hand, it is useful to have Feller property for superprocesses, that is why we will take the approach suggested above.

Let E be a locally compact Hausdorff space with a countable base, and $\mathcal{E} = \mathcal{B}(E)$ σ-algebra of Borel sets. Notice that $(E, \mathcal{E})$ is a standard space whose topology is specified and is not necessarily compact. Let $E_\triangle = E \cup \{\triangle\}$ be the one-point compactification of E, and $\mathcal{E}_\triangle = \mathcal{B}(E_\triangle)$ σ-algebra of Borel sets (we leave E intact if it is already compact). On both E and $E_\triangle$ we consider metric topologies. Recall that $\mathcal{E} = E \cap \mathcal{E}_\triangle$.

As usual, $\mathrm{C}_0(E)$ is the set of all continuous functions on E which vanish at infinity, and $\mathrm{C}(E_\triangle)$ is the set of all continuous functions on $E_\triangle$. $\mathrm{C}_0(E)$ can be regarded as a subset of $\mathrm{C}(E_\triangle)$, and, for every $f \in \mathrm{C}(E_\triangle)$, $f - f(\triangle) \in \mathrm{C}_0(E)$.

Let M be the space of all finite signed measures on $(E, \mathcal{E})$, and $\mathrm{M}_\triangle$ the space of all finite signed measures on $(E_\triangle, \mathcal{E}_\triangle)$. By the Riesz theorem $\mathrm{M} = \mathrm{C}_0(E)^*$, and $\mathrm{M}_\triangle = \mathrm{C}(E_\triangle)^*$. We will be interested in M^+ and $\mathrm{M}_\triangle^+$. Then $\mathrm{M}_\triangle^+$ is a locally compact Hausdorff space with a countable base, with respect to Bernoulli convergence, and $\mathcal{M}_\triangle^+ = \mathcal{B}(\mathrm{M}_\triangle^+)$ (see Remark 2.2.1). However, M^+ is not locally compact (it is Polish in the topology of vague convergence). We can regard M^+ as a subset of $\mathrm{M}_\triangle^+$ in the sense that

$$\mathrm{M}^+ = \{\mu \in \mathrm{M}_\triangle^+ \mid \mu(\{\triangle\}) = 0\} \quad . \tag{2.91}$$

Actually M^+ is a G_δ-subset of $\mathrm{M}_\triangle^+$, and $\mathcal{M}^+ = \mathrm{M}^+ \cap \mathcal{M}_\triangle^+$.

We assume throughout this section that (ξ_t) satisfies a Feller property, i.e., that its Markov semigroup (T_t) is a C_0-semigroup on the Banach space $\mathrm{C}_0(E)$, and that $T_t 1_E \equiv 1$. Then ξ can be regarded as a Hunt process $(\Omega, \mathcal{F}, \mathcal{F}_t, \xi_t, \theta_t, \mathrm{P}^x)$ with infinite life-time, and $\triangle$ as a cemetery (see [4] pp.44-50). Recall that in this situation we extend (T_t) to $\mathrm{C}(E_\triangle)$, by

$$(\tilde{T}_t f)(x) = \begin{cases} (T_t f/E)(x) & x \in E \\ f(\triangle) & x = \triangle \end{cases} \tag{2.92}$$

Notice that $(\tilde{T}_t)$ is a C_0-semigroup on $C(E_\triangle)$. Using Example 2.1.1, we conclude that (T_t) $((\tilde{T}_t))$ is a positive, M $(M_\triangle)$-Pettis integrable semigroup.

We have to restrict our Ψ, too. We assume that $b(x)$ and $c(x)$ are continuous functions which have limits at infinity, i.e., they can be extended to continuous functions $\tilde{b}(x)$, $\tilde{c}(x)$ on $E_\triangle$. We assume also that there exists a kernel $\tilde{n}(x, du)$ from $(E_\triangle, \mathcal{E}_\triangle)$ to $((0, +\infty), \mathcal{B}((0, +\infty)))$ such that, for every $x \in E$, $\tilde{n}(x, du) = n(x, du)$, and, for every $a : (0, +\infty) \longrightarrow [0, +\infty)$, $a(u) \leq u$, and a continuous, the function

$$\mathbb{R}_+ \times E_\triangle \ni (\lambda, x) \mapsto \int_0^{+\infty} a(\lambda u)\tilde{n}(x, du) \tag{2.93}$$

is continuous. If we take $a(u) = u$, we get that 2.53 is a continuous function on $E_\triangle$. If we take $a(u) = 1 - e^{-u}$, we obtain that, for every $f \in C(E_\triangle)^+$, $\tilde{\Psi}$, defined by,

$$(\tilde{\Psi}f)(x) = -\tilde{b}(x)f(x) - \tilde{c}(x)[f(x)]^2 + \int_0^{+\infty}(1 - e^{-f(x)u})\tilde{n}(x, du) \tag{2.94}$$

is continuous on $E_\triangle$. It shows that Ψ $(\tilde{\Psi})$ is a SOLP-operator on $C_0(E)^+$ $(C(E_\triangle)^+)$. Notice that in the case when b, c, and n do not depend on x, all these conditions are satisfied.

By Theorem 2.1.4 there exist (V_t), $V_t : C_0(E)^+ \longrightarrow C_0(E)^+$, and (V_t) a solution of 2.28 for Ψ, and $(\tilde{V}_t)$, $\tilde{V}_t : C(E_\triangle)^+ \longrightarrow C(E_\triangle)^+$ and $(\tilde{V}_t)$ a solution of 2.28 for $\tilde{\Psi}$. By the uniqueness Theorem 2.1.3 and 2.92 both (V_t) and $(\tilde{V}_t)$ are restrictions of the solution of 2.28 for $\tilde{\Psi}$ on $bp\mathcal{E}_\triangle$, and they are related by formula

$$(\tilde{V}_t f)(x) = \begin{cases} (V_t\, f/E)(x) & x \in E \\ f(\triangle) + \int_0^t \tilde{\Psi}(\triangle, (\tilde{V}_s f)(\triangle))ds & x = \triangle \end{cases} \tag{2.95}$$

In particular, it shows that $(\tilde{V}_t f)(\triangle) = (\tilde{V}_t g)(\triangle)$, whenever $f(\triangle) = g(\triangle)$, and $(\tilde{V}_t f)(\triangle) = 0$, whenever $f(\triangle) = 0$. Also, Example 2.1.1 shows that all the functions involved in 2.28 are Bochner integrable, and, therefore, for every $f \in C(E_\triangle)^+$,

$$\tilde{V}_t f = \tilde{T}_t f + (B)\int_{(0,t)} \tilde{T}_s[\tilde{\Psi}(\tilde{V}_{t-s} f)]ds \quad . \tag{2.96}$$

Hence, the mapping $t \mapsto \tilde{V}_t f$ is strongly continuous, and, by 2.94, $t \mapsto V_t f$, for $f \in \mathrm{C}_0(E)^+$, is also strongly continuous.

Using Theorem 2.2.1 we know that there exists one and only one supersemigroup $\tilde{Q}_t(\mu, d\nu)$ which corresponds to the solution of 2.28 on $bp\mathcal{E}_\triangle$. Since $\mathrm{C}(E_\triangle)^+ \subseteq bp\mathcal{E}_\triangle$, it is true that, for every $f \in \mathrm{C}(E_\triangle)^+$,

$$\int_{\mathrm{M}_\triangle^+} \mathrm{e}^{-\langle \nu, f\rangle} \tilde{Q}_t(\mu, d\nu) = \mathrm{e}^{-\langle \mu, \tilde{V}_t f\rangle} \quad . \tag{2.97}$$

Notice that in 2.97 it was important that $\mathrm{M}_\triangle^+$ is equal to M^+ which corresponds to $bp\mathcal{E}_\triangle$, as it was defined in section 2.2.

Consider the functions $F_f : \mathrm{M}_\triangle^+ \longrightarrow \mathbb{R}$, $f \in \mathrm{C}(E_\triangle)^+$, defined by,

$$F_f(\mu) = \mathrm{e}^{-\langle \mu, f\rangle} \quad , \tag{2.98}$$

and the family of functions $Exp(\mathrm{M}_\triangle^+)$, defined as the linear span of the family

$$\left\{ F_f \mid f \in \mathrm{C}(E_\triangle)^+, \ f \text{ strictly positive} \right\} \quad . \tag{2.99}$$

Since f is strictly positive, $F_f \in \mathrm{C}_0(\mathrm{M}_\triangle^+)$ (which makes sense, since $\mathrm{M}_\triangle^+$ is a locally compact space). Since $F_f \cdot F_g = F_{f+g}$, $Exp(\mathrm{M}_\triangle^+)$ is a nonempty algebra, which separates points in $\mathrm{M}_\triangle^+$, and $Exp(\mathrm{M}_\triangle^+) \subseteq \mathrm{C}_0(\mathrm{M}_\triangle^+)$. By the Stone-Weierstraß theorem, the closure of $Exp(\mathrm{M}_\triangle^+)$ is $\mathrm{C}_0(\mathrm{M}_\triangle^+)$.

Recall that, by Theorem 2.3.2 b), $\tilde{T}_t f$ strictly positive implies that $\tilde{V}_t f$ must be strictly positive. If $f \in \mathrm{C}(E_\triangle)^+$ is strictly positive, then, since $E_\triangle$ is compact, there exists a constant $a > 0$, such that, $f \geq a$. $\tilde{T}_t$ is monotone, implies that $\tilde{T}_t f \geq \tilde{T}_t a \equiv a$. Thus, $\tilde{T}_t f$ is strictly positive, and, therefore, $\tilde{V}_t f$ is strictly positive, too. In particular, if $F_f \in Exp(\mathrm{M}_\triangle^+)$, then $F_{\tilde{V}_t f} \in Exp(\mathrm{M}_\triangle^+)$. Using 2.97 we obtain

$$\tilde{Q}_t(F_f) = F_{\tilde{V}_t f} \in Exp(\mathrm{M}_\triangle^+) \quad . \tag{2.100}$$

Since $\tilde{Q}_t$ is a bounded linear operator, and $Exp(\mathrm{M}_\Delta^+)$ is dense in $\mathrm{C}_0(\mathrm{M}_\Delta^+)$, we conclude that

$$\tilde{Q}_t(\mathrm{C}_0(\mathrm{M}_\Delta^+)) \subseteq \mathrm{C}_0(\mathrm{M}_\Delta^+) \quad . \tag{2.101}$$

Recall that $\tilde{V}_t f \longrightarrow f$ strongly, when $t \longrightarrow 0+$. It follows that, for every $\mu \in \mathrm{M}_\Delta^+$,

$$\lim_{t\to 0+} \tilde{Q}_t(F_f)(\mu) = \lim_{t\to 0+} (F_{\tilde{V}_t f})(\mu) = F_f(\mu) \quad . \tag{2.102}$$

Since $(\tilde{Q}_t)$ is a contraction semigroup, $(\tilde{Q}_t F)(\mu) \longrightarrow F(\mu)$, for every $F \in \mathrm{C}_0(\mathrm{M}_\Delta^+)$. In other words, we have proved that $(\tilde{Q}_t)$ satisfies the Feller property. In the case of E compact, and b, c, n constants this result has been proved in Watanabe [56]. Several authors proved the same result later under various assumptions, but essentially Watanabe's method inspired most of these proofs. In this work, we just proved that Watanabe's result is a consequence of our general perturbation theory from section 2.1. and Theorem 2.2.1. However, in section 2.1.we were also inspired by Watanabe's method. Our version of Watanabe's theorem says the following:

Theorem 2.4.1 *If a Markov process ξ with a locally compact Hausdorff state space E with a countable base satisfies Feller condition, and a branching mechanism Ψ satisfies 2.93 and 2.94, then $(\tilde{T}_t, \tilde{\Psi})$-supersemigroup $(\tilde{Q}_t)$ satisfies Feller condition on a locally compact Hausdorff space M_Δ^+, which has a countable base.*

Notice that this result guarantees the existence of a superprocess (X_t) with nice regularity properties. By the basic construction from the theory of Markov processes (see [4], pp.44-50), there exists a Hunt process (X_t) with state space M_Δ^+, which is a superprocess over (ξ_t) on E_Δ with branching mechanism $\tilde{\Psi}$. (X_t) has infinite lifetime, since

$$\tilde{Q}_t 1_{\mathrm{M}_\Delta^+} \equiv 1 \quad .$$

In some sense (X_t) is a superprocess over (ξ_t) and Ψ. But, it has values in $\mathrm{M}_\triangle^+$. We would like to restrict it to M^+ somehow. Notice that there is a (T_t, Ψ)-supersemigroup $Q_t(\mu, d\nu)$ on M^+, when we consider (T_t) and Ψ on $bp\mathcal{E}$. By 2.91 $\mu \in \mathrm{M}^+$ can be regarded as $\mu \in \mathrm{M}_\triangle^+$, with $\mu(\{\triangle\}) = 0$. Thus, for every $\mu \in \mathrm{M}^+$ and $f \in \mathrm{C}_0(E)^+$,

$$\int_{\mathrm{M}_\triangle^+} \mathrm{e}^{-\langle \nu, f\rangle} \tilde{Q}_t(\mu, d\nu) = \mathrm{e}^{-\langle \mu, \tilde{V}_t f\rangle} = \text{by 2.95} =$$

$$= \mathrm{e}^{-\langle \mu, V_t f\rangle} = \int_{\mathrm{M}^+} \mathrm{e}^{-\langle \nu, f\rangle} Q_t(\mu, d\nu) \quad .$$

Since the Laplace transform uniquely determines measure $\tilde{Q}_t(\mu, d\nu)$, it follows that, for every $\mu \in \mathrm{M}^+$ and $t \geq 0$,

$$\tilde{Q}_t(\mu, \mathrm{M}^+) = 1, \text{ and } \tilde{Q}_t(\mu, d\nu)/\mathcal{M}^+ = Q_t(\mu, d\nu) \quad . \tag{2.103}$$

In particular, it shows that, for every $\mu \in \mathrm{M}^+$ and $t \geq 0$,

$$\mathrm{P}^\mu \left[X_t \in \mathrm{M}^+ \right] = 1 \quad . \tag{2.104}$$

Although (X_t) is right continuous, we cannot conclude immediately that the superprocess (X_t) "lives in M^+, if it starts there", since M^+ is not closed in $\mathrm{M}_\triangle^+$. However, the statement is true, as we will show in the following theorem.

Theorem 2.4.2 *For every* $\mu \in \mathrm{M}^+$,

$$\mathrm{P}^\mu \left[X_t \in \mathrm{M}^+, \text{ for every } t \geq 0 \right] = 1 \quad . \tag{2.105}$$

Proof. By our assumptions 2.93 and 2.94, $\bar{\bar{b}}$ is a bounded function, thus, there exists $\beta = \|\bar{\bar{b}}\| \geq 0$. Let $f = 1_\triangle$ be a characteristic function of $\{\triangle\} \subseteq E_\triangle$. Then F, defined by $F(\mu) = \langle \mu, f\rangle = \mu(\triangle)$ is in $\mathcal{M}_\triangle^+$. Using 2.87 we compute

$$(\mathrm{e}^{-\beta t}\tilde{Q}_t)(F)(\mu) = \mathrm{e}^{-\beta t}\langle \mu, \mathrm{E}^x \left[f(\xi_t) \mathrm{e}^{-\int_0^t \bar{\bar{b}}(\xi_s) ds} \right] \rangle \leq$$

$$\leq \langle \mu, \mathrm{E}^x [f(\xi_t)] \rangle = \langle \mu, \tilde{T}_t 1_\triangle \rangle = \langle \mu, 1_\triangle \rangle = F(\mu) \ .$$

By the dominated convergence theorem

$$\lim_{t \to 0+} \mathrm{E}^x \left[f(\xi_t) \mathrm{e}^{-\int_0^t \bar{b}(\xi_s) ds} \right] =$$

$$= \mathrm{E}^x [f(\xi_0)] = (\tilde{T}_t 1_\triangle)(x) = 1_\triangle(x) \ .$$

Notice that $f(\xi_t) = 1_{(\xi_t = \triangle)}$, $\{\triangle\}$ is closed, and $\triangle$ is a cemetery. It shows that

$$\lim_{t \to 0+} f(\xi_t) = f(\xi_0) \ \ (a.e.) \ ,$$

which implies the statement about the limit above. By the dominated convergence theorem

$$\lim_{t \to 0+} (\mathrm{e}^{-\beta t} \tilde{Q}_t)(F)(\mu) = F(\mu) \ \ .$$

Hence, F is β-excessive (relative to (X_t)). It follows, see [4], Theorem 2.12 b), that almost surely the mapping

$$t \mapsto F(X_t) = \langle X_t, 1_\triangle \rangle = X_t(\{\triangle\}) \tag{2.106}$$

is right continuous. By 2.104, we get $X_q(\{\triangle\}) = 0$ a.e., for every $q \in \mathbb{Q}^+$. Using 2.106 we obtain that almost surely the mapping

$$t \mapsto X_t(\{\triangle\})$$

is identically zero, which proves the theorem.

Q.E.D.

The fact that (T_t) satisfies Feller condition, i.e., that (T_t) is a C_0-semigroup on $\mathrm{C}_0(E)$, and $(\tilde{T}_t)$ is a C_0-semigroup on $\mathrm{C}(E_\triangle)$, introduces another powerful "tool" into consideration. It is the infinitesimal generator. Let A be the infinitesimal generator of (T_t), defined on its domain $\mathcal{D}(A) \subseteq \mathrm{C}_0(E)$, and $\tilde{A}$ the infinitesimal generator of

$(\tilde{T}_t)$, defined on its domain $\mathcal{D}(\tilde{A}) \subseteq \mathrm{C}(E_\triangle)$. Recall that $f \in \mathcal{D}(\tilde{A})$ if and only if $f - f(\triangle) \in \mathcal{D}(A)$, and

$$(\tilde{A}f)(x) = \begin{cases} A(f - f(\triangle))/E(x) & x \in E \\ 0 & x = \triangle \end{cases} \tag{2.107}$$

Denote the infinitesimal generator of $(\tilde{Q}_t)$ by $\mathcal{G}$. Consider the family of functions $Exp(\mathrm{M}^+_\triangle, \mathcal{D}(\tilde{A}))$, defined as the linear span of the family

$$\left\{ F_f \mid f \in \mathrm{C}(E_\triangle)^+ \cap \mathcal{D}(\tilde{A}),\ f \text{ strictly positive} \right\} . \tag{2.108}$$

In [56] Theorem 2.4 and Lemma 2.2, Watanabe proved that (in the case of constant coefficients b, c, n) $Exp(\mathrm{M}^+_\triangle, \mathcal{D}(\tilde{A}))$ is a core for $\mathcal{G}$, and he also computed $\mathcal{G}(F_f)$. Since exactly the same proof works in our case without any difficulties, there is no need to repeat the proof here. We conclude that the following statements are true.

$$Exp(\mathrm{M}^+_\triangle, \mathcal{D}(\tilde{A})) \text{ is a core for } \mathcal{G} . \tag{2.109}$$

For every $f \in \mathrm{C}(E_\triangle)^+ \cap \mathcal{D}(\tilde{A})$, f strictly positive, and for every $t \geq 0$,

$$\tilde{V}_t f \in \mathrm{C}(E_\triangle)^+ \cap \mathcal{D}(\tilde{A}),\ \tilde{V}_t f \text{ is strictly positive.} \tag{2.110}$$

For every $f \in \mathrm{C}(E_\triangle)^+ \cap \mathcal{D}(\tilde{A})$, f strictly positive,

$$\mathcal{G}(F_f)(\mu) = -\mathrm{e}^{-\langle \mu, f \rangle} \cdot \langle \mu, \tilde{A}f + \tilde{\Psi}f \rangle =$$

$$= -(F_f)(\mu) \cdot \langle\, \mu,\ \tilde{A}f - \tilde{b}f - \tilde{c}f^2 + \int_0^{+\infty} (1 - \mathrm{e}^{-fu}) \tilde{n}(x, du) \rangle \ . \tag{2.111}$$

For the "martingale problem"-type characterization of the infinitesimal generator see [28] pp.338 and 354.

It is now an easy Corollary of these results, to give a complete characterization of infinitesimal generators of superprocesses over finite Markov chains with standard transition matrix. Let (ξ_t) be a Markov chain with state space $\{1, 2, \ldots, n\}$ and a

standard transition matrix (T_t) with infinitesimal generator A, described by 2.79. Notice that our state space is already compact, so there is no need to use one-point compactification. Our branching mechanism is determined by b, c, and n, where now $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$, $c = (c_1, \ldots, c_n) \in \mathbb{R}^n_+$, and $n = (n_1(du), \ldots, n_n(du))$ where each measure $n_i(du)$ satisfies condition

$$\int_0^{+\infty} u n_i(du) < +\infty \quad . \tag{2.112}$$

There are no other restrictions on Ψ.

In this case $A = \tilde{A}$ and $\mathcal{D}(A) = \mathbb{R}^n$. Notice that, for every $f \in \mathbb{R}^n_+$, and a function $F_f : \mathbb{R}^n_+ \longrightarrow \mathbb{R}_+$,

$$(F_f)(\mu) \cdot \langle \mu, Af \rangle = \mathrm{e}^{-\sum_{i=1}^n f_i \mu_i} \cdot \sum_{j=1}^n \mu_j \left(\sum_{k=1}^n a_{jk} f_k \right) =$$

$$= \sum_{j=1}^n \mu_j \left(\sum_{k=1}^n a_{jk} f_k \mathrm{e}^{-\sum_{i=1}^n f_i \mu_i} \right) = \sum_{j=1}^n \mu_j \left[\sum_{k=1}^n a_{jk} \left(-\frac{\partial F_f}{\partial \mu_k} \right) \right] =$$

$$= -\langle \mu, \ A \cdot \nabla F_f \rangle \quad .$$

Similarly we obtain

$$(F_f)(\mu) \cdot \langle \mu, bf \rangle = -\sum_{j=1}^n \mu_j b_j \frac{\partial F_f}{\partial \mu_j} = -\langle \mu, \ b \bullet \nabla F_f \rangle$$

and

$$(F_f)(\mu) \cdot \langle \mu, cf^2 \rangle = \sum_{j=1}^n \mu_j c_j \frac{\partial^2 F_f}{\partial \mu_j^2} \quad .$$

The last term in 2.111 can be rewritten as,

$$(F_f)(\mu) \cdot \langle \mu, \int_0^{+\infty} (1 - \mathrm{e}^{-fu}) n(i, du) \rangle =$$

$$= -\sum_{i=1}^n \mu_i \cdot \int_0^{+\infty} \left(F_f(\mu + u\varepsilon_i) - F_f(\mu) \right) n_i(du)$$

where $\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$. Using Theorem 2.2.2, results about Kac semigroup, and results in this section (recall that exponential functions form a core for

$\mathcal{G}$; and $\mathcal{G}$ is determined on its core), we can summarize properties of superprocesses over Markov chains in the following theorem.

Theorem 2.4.3 *A superprocess (X_t) over Markov chain (ξ_t) with branching mechanism Ψ is a Markov process which satisfies Feller condition, and is actually a Hunt process with infinite life-time whose state space is $\mathbb{R}^n_+$. The Laplace transform of a supersemigroup $(Q_t(\mu, d\nu))$ is given by*

$$\int_{\mathbb{R}^n_+} \mathrm{e}^{-\sum_{i=1}^n f_i \nu_i} Q_t((\mu_1, \ldots, \mu_n), (d\nu_1, \ldots, d\nu_n)) = \mathrm{e}^{-\sum_{i=1}^n \mu_i (V_t f)_i} , \tag{2.113}$$

where

$$(V_t f)_i = \sum_{j=1}^n p_{ij}(t) f_j +$$

$$+ \int_0^t \sum_{j=1}^n p_{ij}(t-s) \left[-b_j (V_s f)_j - c_j (V_s f)_j^2 + \int_0^{+\infty} (1 - \mathrm{e}^{-(V_s f)_j u}) n_j(du) \right] ds =$$

$$= f_i + \int_0^t ds \sum_{j=1}^n p_{ij}(t-s) \cdot \tag{2.114}$$

$$\cdot \left[-b_j (V_s f)_j - c_j (V_s f)_j^2 + \int_0^{+\infty} (1 - \mathrm{e}^{-(V_s f)_j u}) n_j(du) + \sum_{k=1}^n a_{jk} (V_s f)_k \right] .$$

The infinitesimal generator $\mathcal{G}$ of (X_t) is given by

$$(\mathcal{G}F)(\mu) = \tag{2.115}$$

$$= \sum_{j=1}^n \mu_j \cdot \left[c_j \frac{\partial^2 F}{\partial \mu_j^2} - b_j \frac{\partial F}{\partial \mu_j} + (A \cdot \nabla F)_j + \int_0^{+\infty} [F(\mu + u\varepsilon_j) - F(\mu)] n_j(du) \right] .$$

In the special case, when $\Psi = -b\lambda$, (V_t) is the Kac semigroup and the infinitesimal generator is

$$(\mathcal{G}F)(\mu) = \langle \mu, \ A \cdot \nabla F - b \bullet \nabla F \rangle \ . \tag{2.116}$$

Let us show, in the example below, that even on the level of Markov chains, the superprocess (X_t) over (ξ_t) does not necessarily satisfy Feller condition, if (ξ_t) does not satisfy Feller condition.

Example 2.4.1 We consider $n = 2$, and (T_t) given by 2.80, i.e.,

$$T_t = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} .$$

Let $(\Psi f)_i = -bf_i$, $i = 1, 2$; where $b \in \mathbb{R}$. Using results about Kac semigroup it is easy to check that

$$(V_t f)_i = \mathrm{e}^{-bt}(T_t f)_i = \mathrm{e}^{-bt} f_2 \ , \tag{2.117}$$

and

$$Q_t((\mu_1, \mu_2); (d\nu_1, d\nu_2)) = \Delta_{(0,(\mu_1+\mu_2)\mathrm{e}^{-bt})}(d\nu_1, d\nu_2) \ , \tag{2.118}$$

which gives a superprocess $X_t = (X_t^1, X_t^2)$ to be

$$(X_t^1, X_t^2) = (0, (X_0^1 + X_0^2)\mathrm{e}^{-bt}) \ , \tag{2.119}$$

and it is obvious that such a Markov process does not satisfy Feller condition.$\diamondsuit$

2.5 Branching Property

In this section we will present some easy consequences of the fact that superprocesses satisfy a branching property. Branching property in this form, i.e., in the continuous state space has been introduced by M. Jiřina [36].

Notice that the additive structure of M^+ enables us to introduce a convolution operation for measures on $(\mathrm{M}^+, \mathcal{M}^+)$. If Q_1 and Q_2 are probability measures on $(\mathrm{M}^+, \mathcal{M}^+)$, then $Q_1 \times Q_2$ is a probability measure on $(\mathrm{M}^+ \times \mathrm{M}^+, \mathcal{M}^+ \otimes \mathcal{M}^+)$. Since $(\mu, \nu) \mapsto \mu + \nu$ is $\mathcal{M}^+ \otimes \mathcal{M}^+ / \mathcal{M}^+$ measurable, there is a unique measure, say $Q_1 * Q_2$ on $(\mathrm{M}^+, \mathcal{M}^+)$, defined by

$$(Q_1 * Q_2)(\mathbf{H}) = \int_{\mathrm{M}^+ \times \mathrm{M}^+} 1_{\mathbf{H}}(\mu + \nu)(Q_1 \times Q_2)(d\mu, d\nu) \ , \tag{2.120}$$

where $\mathbf{H} \in \mathcal{M}^+$. As usual, we will call the measure $Q_1 * Q_2$ *the convolution of measures* Q_1 *and* Q_2. Notice that it is easy to check that the Laplace transform of

$Q_1 * Q_2$ is the product of Laplace transforms of Q_1 and Q_2. Also, if X_1 and X_2 are independent M^+-valued random variables with distributions Q_1 and Q_2, then $X_1 + X_2$ has a distribution $Q_1 * Q_2$.

Using the fact that $\langle \mu_1 + \mu_2, V_t f \rangle = \langle \mu_1, V_t f \rangle + \langle \mu_2, V_t f \rangle$ we conclude that a Dawson-Watanabe supersemigroup $(Q_t(\mu, d\nu))$ satisfies the property

$$Q_t(\mu_1 + \mu_2, d\nu) = Q_t(\mu_1, d\nu) * Q_t(\mu_2, d\nu) \quad . \tag{2.121}$$

The consequence is that, if (X_t^1) and (X_t^2) are two independent copies of Dawson-Watanabe superprocesses over (ξ_t) with branching mechanism Ψ, where $X_0^1 = \mu_1$ and $X_0^2 = \mu_2$ then

$$X_t = X_t^1 + X_t^2 \text{ is a Dawson} - \text{Watanabe superprocess, } X_0 = \mu_1 + \mu_2. \tag{2.122}$$

Property 2.121 (and 2.122 as well) is a *branching property*, i.e., every Dawson-Watanabe superprocess is a *branching process.*

Let us fix $\mu \in \mathrm{M}^+$. Then, for every $a \geq 0$, $a\mu \in \mathrm{M}^+$. The branching property 2.121 implies that a family

$$\{\, Q_t(a\mu, d\nu)\,;\ a \geq 0 \,\} \tag{2.123}$$

is a convolution semigroup (notice that t and μ are fixed, and $a \in \mathbb{R}_+$ is a parameter). It shows that, once we know $Q_t(\delta_x, d\nu)$ for every $x \in E$, we can use 2.123 to find $Q_t(a\delta_x, d\nu)$, and then 2.121 to find $Q_t(\mu, d\nu)$ for every μ of the form

$$\mu = \sum_{i=1}^{n} a_i \delta_{x_i} \quad . \tag{2.124}$$

Let us also mention here that in the case of standard space $(E, \mathcal{E})$ and corresponding space $(\mathrm{M}^+, \mathcal{M}^+)$, it has been proven in [18], pp.264-265, that, if $\mathcal{L}_{Q_n}(f) \longrightarrow \mathcal{L}(f)$ as $n \to +\infty$, for every $f \in bp\mathcal{E}$, then there exists a subprobability measure Q, such

that $\mathcal{L}_Q = \mathcal{L}$, where $\mathcal{L}$ denotes the Laplace transform. In particular, if $\langle \mu_n, V_t f \rangle \longrightarrow \langle \mu, V_t f \rangle$, for every $f \in bp\mathcal{E}$, then

$$\mathcal{L}_{Q_t(\mu_n, d\nu)} \longrightarrow \mathcal{L}_{Q_t(\mu, d\nu)} \quad . \tag{2.125}$$

CHAPTER 3
ONE-DIMENSIONAL CASE

In this chapter we describe the simplest possible superprocesses, i.e., the superprocesses over Markov chain with one state space. Since these superprocesses are one-dimensional branching processes, they are recruited among processes which were studied before. We use one-dimensional superprocesses then to describe general superprocesses over deterministic Markov processes which stay in the initial state forever (but the state space of the Markov process is arbitrary). In this way we obtain the "behaviour" of the superprocess which is "influenced" by the branching mechanism part, only. Then, in the next chapter, we show how in some cases of Markov processes we can get a description of the corresponding superprocesses by using the results of this chapter. It should be emphasized that our main tool is the analysis of the supertransition function.

3.1 Supersemigroups and Infinitesimal Generators

We restrict our attention to the one-dimensional case in this and the following section. Hence, we consider the case when $E = \{1\}$. Then a branching mechanism Ψ depends on λ only, i.e.,

$$\Psi(\lambda) = -b\lambda - c\lambda^2 + \int_0^{+\infty} (1 - e^{-\lambda u}) n(du) \quad , \tag{3.1}$$

where $b \in \mathbb{R}$, $c \in \mathbb{R}_+$, and $\int_0^{+\infty} un(du) < +\infty$. In this case we do not have much choice for (ξ_t), i.e., $\xi_t \equiv \xi_0 \equiv 1$, $T_t = Id_t$, and $A = 0$. Also $\mathbf{M}^+ = \mathbb{R}_+$, and μ and f are nonnegative real numbers. Using now the results from section 2.4. in this special case, we conclude that a Dawson-Watanabe superprocess over (ξ_t) with branching

mechanism Ψ is an $\mathbb{R}_+$-valued Hunt process (X_t) whose infinitesimal generator $\mathcal{G}$ is given by

$$(\mathcal{G}F)(\mu) = \mu\left\{c\frac{d^2F}{d\mu^2} - b\frac{dF}{d\mu} + \int_0^{+\infty}[F(\mu+u) - F(\mu)]n(du)\right\} \quad . \tag{3.2}$$

The supersemigroup $(Q_t(\mu, d\nu);\ \mu \geq 0)$ is a convolution semigroup with respect to μ, whose Laplace transform is determined by $(V_t;\ t \geq 0)$, where, for every $f \geq 0$,

$$(V_t f) = f + \int_0^t \left[-b(V_s f) - c(V_s f)^2 + \int_0^{+\infty}(1 - e^{-(V_s f)u})n(du)\right] ds \ . \tag{3.3}$$

Since we know that $V_t f$ exists, 3.3 shows that $t \mapsto V_t f$ is a continuous function, and, again by 3.3, $t \mapsto V_t f$ is a continuosly differentiable function on $\mathbb{R}_+$, which satisfies the differential equation

$$\begin{cases} \frac{d(V_t f)}{dt} = -b(V_t f) - c(V_t f)^2 + \int_0^{+\infty}(1 - e^{-(V_t f)u})n(du) \\ (V_0 f) = f \end{cases} \tag{3.4}$$

Let us consider some cases in which we can explicitly solve 3.4.

Case $\Psi(\lambda) = -b\lambda - c\lambda^2;\ c > 0, b \in \mathbb{R}$.

In this case the differential equation 3.4 is a Riccati type equation

$$\begin{cases} y' = -by - cy^2 \\ y(0) = f \end{cases} \tag{3.5}$$

which can be easily solved, by separation of variables, to give

$$V_t f = \frac{f}{e^{bt} + fc\frac{e^{bt}-1}{b}} \quad , \tag{3.6}$$

where $(e^{bt} - 1)/b$ is defined by

$$\frac{e^{bt}-1}{b} = \begin{cases} \frac{e^{bt}-1}{b} & \text{if } b \neq 0 \\ \lim_{b\to 0}\frac{e^{bt}-1}{b} = t & \text{if } b = 0 \end{cases} \tag{3.7}$$

Hence, in the case when $b = 0$ we have

$$V_t f = \frac{f}{1 + fct} \quad . \tag{3.8}$$

Using the fact that $V_t f$ describes the Laplace transform of $Q_t(\mu, d\nu)$, and by following simple computation

$$-\langle \mu, V_t f \rangle = -\frac{\mu f}{e^{bt} + f c\frac{e^{bt}-1}{b}} =$$

$$= -\frac{\mu}{c\frac{e^{bt}-1}{b}} + \frac{\mu}{c\frac{e^{bt}-1}{b}} \cdot \frac{\frac{e^{bt}}{c\frac{e^{bt}-1}{b}}}{f + \frac{e^{bt}}{c\frac{e^{bt}-1}{b}}}$$

we obtain that (recall that μ and ν are points in $\mathbb{R}_+$ here),

$$Q_t(\mu, d\nu) = e^{-\frac{\mu}{c\frac{e^{bt}-1}{b}}} \left[\delta_0(d\nu) + \sum_{n=1}^{\infty} \frac{G_{\mu,t}^{n*}(d\nu)}{n!} \right] , \tag{3.9}$$

where

$$G_{\mu,t} \sim \frac{\mu}{c\frac{e^{bt}-1}{b}} \mathrm{EXP}\left(\frac{e^{bt}}{c\frac{e^{bt}-1}{b}} \right) , \tag{3.10}$$

where, as usual, $\mathrm{EXP}(\lambda)$ denotes the exponential distribution with parameter $\lambda > 0$. The infinitesimal generator of this process is (by 3.2)

$$c\mu \frac{d^2}{d\mu^2} - b\mu \frac{d}{d\mu} . \tag{3.11}$$

If we compare these results to Corollary 2 of [44] we get the following simple consequence:

Corollary 3.1.1 (X_t) *is a one-dimensional superprocess with continuous paths if and only if it is a one-dimensional continuous branching diffusion. In this case* (X_t) *has a transition probability given by 3.9 and infinitesimal generator given by 3.11.*

Remark 3.1.1 It should be noted that these processes have been studied before, and it was observed that they represent superprocesses. Here we have shown how these results can be derived in very simple and elementary way following our approach. Also, the transition function is not usually mentioned in the form 3.9 in the literature, and in the literature on superprocesses is usually not mentioned at all. However, in our line of approach we will rely heavily on a transition function.

The diffusions with infinitesimal generator 3.11 were studied first by W. Feller in 1939 and especially in 1951 [26]. At that time continuous state branching processes were not introduced yet, and Feller has shown that these diffusions are the limits of discrete parameter branching processes. He treated them as $(0, +\infty)$-valued diffusions with 0 as an absorbing boundary. We include 0 in the state space, and because of that we have a singular first term in 3.9. Feller computed also a density part of the transition function 3.9 in terms of Bessel functions. The case $c = 1/2$, $b = 0$ is treated in [42] pp.100-102, again from the diffusion point of view. In this case the transition function has a nice form

$$Q_t(\mu, d\nu) = \mathrm{e}^{-\frac{2\mu}{t}}\delta_0(d\nu) + 2\mathrm{e}^{-\frac{2(\mu+\nu)}{t}}\frac{\sqrt{\mu\nu}}{t}I_1(4\frac{\sqrt{\mu\nu}}{t})\frac{d\nu}{\nu}\ , \tag{3.12}$$

where I_1 is the modified Bessel function of the first kind. We will not use 3.12 any more. It is mentioned here just to complete the description of the processes above.

In 1958 M. Jiřina introduced continuous state branching processes in [36]. In 1967 J. Lamperti covered the one-dimensional case in [44] and [45]. He gave us formula 3.6 (for the first time as far as we can tell). A nice brief account of these processes from the point of view of branching processes is given in [1] pp.257-261. Several authors on superprocesses, like D. Dawson, S. Roelly-Coppoletta, etc., emphasized that these diffusions (or some of them) are special cases of superprocesses. Almost exclusively they deal with the infinitesimal generators.□

Case $\Psi(\lambda) = m(\,1 - \mathrm{e}^{-a\lambda}\,);\ m > 0,\ a > 0.$

This is the case when $n(du) = m\delta_a$. Again, the differential equation 3.4 is easy to solve, since it is in a form

$$\begin{cases} y' = m(\,1 - \mathrm{e}^{-ay}\,) \\ y(0) = f \end{cases} \tag{3.13}$$

Separation of variables gives the solution

$$V_t f = \frac{1}{a} \ln \left[e^{amt}(e^{af} - 1) + 1 \right] \quad . \tag{3.14}$$

Simple computation shows that

$$e^{-\langle \mu, V_t f \rangle} = \frac{1}{[e^{amt}(e^{af} - 1) + 1]^{\mu/a}} = e^{-\mu m t} e^{-\mu f} [1 + e^{-af}(e^{-amt} - 1)]^{-\mu/a} \quad .$$

Since $e^{-af}(e^{-amt} - 1) \in (-1, 1)$ we can apply Newton's binomial formula to get

$$e^{-\langle \mu, V_t f \rangle} = e^{-\mu m t} e^{-\mu f} \cdot \sum_{n=0}^{\infty} \binom{-\frac{\mu}{a}}{n} \left[e^{-af}(e^{-amt} - 1) \right]^n =$$

$$= \sum_{n=0}^{\infty} \binom{-\frac{\mu}{a}}{n} e^{-\mu m t}(e^{-amt} - 1)^n e^{-f(\mu + na)} \quad .$$

It follows that the supertransition function is given by

$$Q_t(\mu, d\nu) = \sum_{n=0}^{\infty} \binom{-\frac{\mu}{a}}{n} e^{-\mu m t}(e^{-amt} - 1)^n \delta_{\mu + na}(d\nu) \quad . \tag{3.15}$$

The infinitesimal generator of this process is (by 3.2)

$$(\mathcal{G}F)(\mu) = m\mu \left[F(\mu + a) - F(\mu) \right] \quad . \tag{3.16}$$

Remark 3.1.2 There are descriptions of these processes in [44] and [1], but they are treated mostly as time-changed Poisson processes (see more about it in the next section). However, we are not aware of formulas 3.14 and 3.15 in the literature. Here they follow easily from the general approach on superprocesses. Let us also mention that in the case $a = m = 1$, and $\mu = 1$, the supertransition function 3.15 is given by

$$Q_t(1, d\nu) = \sum_{n=0}^{\infty} e^{-t}(1 - e^{-t})^n \delta_{1+n}(d\nu) \quad , \tag{3.17}$$

which is a *geometric distribution.*□

It is a good place to clarify certain facts about simple transformations of superprocesses. Let (X_t) be a Dawson-Watanabe superprocess over a Markov process (ξ_t) on $(E, \mathcal{E})$ with branching mechanism $\Psi(x, \lambda)$. We consider general LCCB space $(E, \mathcal{E})$. Let $f \geq 0$ be a bounded, nonnegative, $\mathcal{E}$-measurable function. Then the stochastic process (Y_t), defined by

$$Y_t = \langle X_t, f \rangle \tag{3.18}$$

is an $\mathbb{R}_+$-valued stochastic process. One may hope that (Y_t) is a one-dimensional superprocess somehow related to Ψ. This is particularly so in the case when $f \equiv 1$, i.e., (Y_t) is the, so called, *total-mass process*. However, this is not always the case. The following two examples indicate typical difficulties.

Example 3.1.1 Let (ξ_t) be the uniform motion to the right on the real line, and $\Psi(x, \lambda) = -\lambda$. Let $f = 1_{[1,+\infty)}$. Using Theorem 2.3.1 and formula 2.85 we obtain

$$Y_t = \langle X_t, f \rangle = \int_{\mathbb{R}} X_0(dx)\, \mathrm{E}^x \left[f(\xi_t) \mathrm{e}^{-\int_0^t 1(\xi_u) du} \right] =$$

$$= \int_{\mathbb{R}} X_0(dx) (\mathrm{e}^{-t} f(x+t)) = \mathrm{e}^{-t} \cdot X_0([1-t, +\infty)) \quad .$$

Hence, for every $X_0 \equiv \mu \neq 0$, where $\mathrm{supp}(\mu) \subseteq (-\infty, 0)$, we will have that $Y_t \equiv 0$, for $0 \leq t \leq 1$, and $Y_t > 0$, for t large enough. But, there is no one-dimensional superprocess with such a behaviour. For every one-dimensional superprocess it is true that if it starts at zero, it stays there forever, since the Laplace transform of $Q_t(0, d\nu)$ is identically one.$\diamondsuit$

Example 3.1.2 Let (ξ_t) be again the uniform motion to the right on the real line, and $\Psi(x, \lambda) = -b(x)\lambda$, where

$$b(x) = \begin{cases} -1\,; & x \leq -1 \\ x\,; & x \in (-1, 1) \\ 1\,; & x \geq 1 \end{cases}$$

Let $f \equiv 1$. Using Theorem 2.3.1 and formula 2.85 again, we obtain

$$Y_t = \langle X_t, 1 \rangle = \langle X_0, \mathrm{E}^x \left[\mathrm{e}^{-\int_0^t b(\xi_u) du} \right] \rangle \ .$$

If $X_0 \equiv \delta_a$, $a \in \mathbb{R}$, then

$$Y_t = \mathrm{E}^a \left[\mathrm{e}^{-\int_0^t b(\xi_u) du} \right] = \mathrm{e}^{-\int_0^t b(a+u) du} \ .$$

It shows that Y_t is a deterministic process, and for $a = 0$ it is equal to

$$Y_t = \begin{cases} \mathrm{e}^{-t^2/2} & ; \ t \in [0,1] \\ \mathrm{e}^{-1/2} + (t-1) & ; \ t \geq 1 \end{cases} \tag{3.19}$$

However, there is no such one-dimensional superprocess. As we can see from 3.2 the only deterministic one-dimensional superprocesses are the processes of the form $X_t = X_0 \mathrm{e}^{-bt}$.$\diamondsuit$

Notice that in the first example the coefficients of Ψ are constant, but $f \neq 1$, while in the second the coefficients of Ψ are not constant. Several authors considered constant coefficients and $f \equiv 1$ (see, for example [52]). In this case $\langle X_t, 1 \rangle$ "behaves" as we would initially expect. For the sake of completeness let us prove this result.

Proposition 3.1.1 *Let $(E, \mathcal{E})$ be a standard space, and (X_t) a Dawson-Watanabe superprocess over (ξ_t) with state space $(E, \mathcal{E})$, and with branching mechanism Ψ, where Ψ has constant coefficients, i.e., $b(x) \equiv b$, $c(x) \equiv c$, and $n(x) \equiv n$. Then $\langle X_t, 1 \rangle$ is a one-dimensional superprocess with the same branching mechanism Ψ.*

Proof. Notice that the statement makes sense, since Ψ can be regarded as a branching mechanism over $\{1\}$, as well. Let (V_t) be a nonlinear semigroup which corresponds to (X_t). We claim that for every $\lambda \geq 0$, $V_t(\lambda 1)(x)$ is a constant function. Recall, by construction, that V_t is obtained as a limit of

$$V_t^{n+1}(\lambda 1)(x) = T_t(\lambda 1)(x) + \int_0^t T_{t-s}[\Psi(V_s^n(\lambda 1))](x) ds \ ,$$

where $V_t^0 \equiv 0$, and $T_t(\lambda 1) = \lambda T_t 1 \equiv \lambda$. Assuming that $V_s^n(\lambda 1)$ is a constant, we get that $\Psi(V_s^n(\lambda 1))$ is a constant function, since Ψ has constant coefficients. Then $T_{t-s}[\Psi(V_s^n(\lambda 1))]$ is a constant, too, since $T_{t-s}1 \equiv 1$. It follows that $V_t^{n+1}(\lambda 1)$ is a constant function. By mathematical induction we obtain that $x \mapsto V_t(\lambda 1)(x)$ is a constant function, for every $\lambda \geq 0$, and it satisfies

$$V_t(\lambda 1) = \lambda + \int_0^t \Psi[V_s(\lambda 1)]ds \quad .$$

If we define $W_t\lambda = V_t(\lambda 1)$, then W_t satisfies 3.3 (by the equality above). Since, for $\lambda \geq 0$ and $X_0 \equiv \mu$,

$$\int_{M^+} e^{-\lambda\langle X_t,1\rangle} dP = \int_{M^+} e^{-\langle X_t,\lambda 1\rangle} dP = e^{-\langle \mu, V_t(\lambda 1)\rangle} =$$

$$= e^{-\mu(E)\cdot V_t(\lambda 1)} = e^{-\langle X_0,1\rangle \cdot V_t(\lambda 1)} = e^{-\langle X_0,1\rangle \cdot W_t\lambda} \quad ,$$

we have shown that $\langle X_t, 1\rangle$ is a one-dimensional superprocess with branching mechanism Ψ.

Q.E.D.

Remark 3.1.3 It is possible to study $\langle X_t, f\rangle$ also by using martingale theory approach. We will not enter this subject. Let us only mention some references like [10] and [52]. Several results on semimartingale properties of

$$\exp(-\langle X_t,\ f\rangle) \quad ,$$

where f is in the domain of the infinitesimal generator of (ξ_t), are given in Chapter 5 of [10].□

3.2 Time Change and Zero-One Law

We have seen in the previous section that one-dimensional superprocesses are actually one-dimensional branching processes studied by J. Lamperti in the late sixties. J. Lamperti has also shown that these processes can be obtained as a time

change of Lévy processes. Here we will emphasize Lamperti's result and show that related to this is an interesting zero-one law for Brownian motion, which in some sense complement Engelbert-Schmidt zero-one law (see [23]).

Let $\Psi(\lambda)$ be given by 3.1, and (L_t^Ψ) a corresponding Lévy process given by 2.76. Then we have

$$L_t^\Psi = -bt + \sqrt{c}\, B_t + J_t \quad , \tag{3.20}$$

where (B_t) and (J_t) are independent copies of standard Brownian motion and pure jump process with *positive* jumps, respectively (see 2.76 and [35] for details). In particular, (L_t^Ψ) is a strong Markov process with the infinitesimal generator given by

$$-b\frac{d}{dx} + c\frac{d^2}{dx^2} + \int_0^{+\infty} [\, F(x+u) - F(x) \,]\, n(du) \quad . \tag{3.21}$$

Consider now only the cases when $L_0^\Psi \equiv a \geq 0$, and let $(\tilde{L}_t^\Psi)$ be the process (L_t^Ψ) stopped when it reaches zero. Then $(\tilde{L}_t^\Psi)$ is a Markov process with state space $\mathbb{R}_+$, whose infinitesimal generator is given by 3.21 for positive x, and is equal to zero for $x = 0$.

Consider the additive functional (A_t^Ψ) defined by

$$A_t^\Psi = \int_0^t \frac{ds}{\tilde{L}_s^\Psi} \quad . \tag{3.22}$$

Notice that (A_t^Ψ) is a continuous additive functional, except maybe at the point $\tau_0 = \inf\{t > 0 : \tilde{L}_t^\Psi = 0\}$, which is equal to $\inf\{t > 0 : L_t^\Psi = 0\}$. Obviously $A_{\tau_0+}^\Psi = +\infty$, but it is possible that $A_{\tau_0-}^\Psi < +\infty$. We will see later that it is so indeed in many cases. Therefore, we define $A_{\tau_0}^\Psi$ to be equal to $A_{\tau_0-}^\Psi$. Hence, $t \mapsto A_t^\Psi$ is continuous and strictly increasing on $[0, \tau_0]$, and the same is true for its inverse. We define the inverse of $t \mapsto A_t^\Psi$ to be equal to τ_0, for every $s \geq A_{\tau_0}^\Psi$. Once this is settled the standard results on time-changed process apply. Therefore, let (X_t^Ψ) be a process obtained from $(\tilde{L}_t^\Psi)$ by a random time change, corresponding to the additive

functional (A_t^Ψ). Using Theorem 10.12 from [16] we obtain that (X_t^Ψ) is a Markov process whose infinitesimal generator is given by 3.2, i.e., (X_t^Ψ) is a one-dimensional superprocess described in section 3.1.

Remark 3.2.1 The result above belongs to J. Lamperti. It was announced in [44], but as far as we know its proof was not published in author's later papers. Of course, the proof is more or less standard application of random time change, and it was outlined in [44]. One only has to be more careful with the possible discontinuity at τ_0, as we explained above.□

Let us investigate some cases in which $A^\Psi_{\tau_0-}$ is finite. Consider the standard Brownian motion (B_t) on the real line. Let $\tau_0 = \inf\{t > 0 : B_t = 0\}$. A consequence of the following theorem is that $A^\Psi_{\tau_0-}$ is finite if $\Psi(\lambda) = -\lambda^2$.

Theorem 3.2.1 (Zero-One Law) For every $a > 0$

$$\mathrm{P}^a\left[\int_0^{\tau_0} \frac{ds}{B_s^\alpha} < +\infty\right] = \begin{cases} 1 & ; \text{ if } 0 < \alpha < 2 \\ 0 & ; \text{ if } \alpha \geq 2 \end{cases} \tag{3.23}$$

Remark 3.2.2 Notice that Theorem 3.2.1 is an interesting result in itself. Recall that the Engelbert-Schmidt $0-1$ law implies (see [23])

$$\mathrm{P}^0\left[\int_0^t \frac{ds}{|B_s|^\alpha} < +\infty\,; \forall t \geq 0\right] = \begin{cases} 1 & ; \text{ if } 0 < \alpha < 1 \\ 0 & ; \text{ if } \alpha \geq 1 \end{cases}$$

which says something about the behaviour of the Brownian trajectory when it leaves zero (or on the right hand side of zero). Our result 3.23 complements Engelbert-Schmidt's result in the sense that it compares what is happening when Brownian motion enters zero (or on the left hand side of zero). Obviously, the behaviour is different. Notice also that our result is not the consequence of general criteria in [23], since these result only show that for $\alpha \geq 1$, and $a > 0$,

$$\mathrm{P}^a\left[\int_0^t \frac{ds}{|B_s|^\alpha} < +\infty\,; \forall t \geq 0\right] = 0 \quad ,$$

which does not imply anything about the integral up to τ_0.□

Proof. Let $\tau_b = \inf\{t > 0 : B_t = b\}$ and, for $n \geq 1$, $a_n = a/n$. Then we have, since $\mathrm{P}^a(B_0 = a) = 1$,

$$\int_0^{\tau_0} \frac{ds}{B_s^\alpha} = \sum_{n=1}^{\infty} \int_{\tau_{a_n}}^{\tau_{a_{n+1}}} \frac{ds.}{B_s^\alpha} \quad \mathrm{P}^a - \text{a.e.},$$

and the sum on the right hand side is the sum of nonnegative, independent (with respect to P^a) random variables. Hence, it is P^a – a.e. finite or P^a – a.e. infinite.

Consider $0 < \alpha < 2$ first. If $s \leq \tau_{a_{n+1}}$ then $B_s \geq a_{n+1}$ P^a – a.e. It follows that

$$\int_0^{\tau_0} \frac{ds}{B_s^\alpha} \leq \sum_{n=1}^{\infty} \frac{1}{a_{n+1}^\alpha} (\tau_{a_{n+1}} - \tau_{a_n}) \quad \mathrm{P}^a - \text{a.e.}$$

If we denote $(1/a_{n+1}^\alpha)(\tau_{a_{n+1}} - \tau_{a_n})$ by Z_n, then $\{ Z_n;\ n \geq 1 \}$ is the sequence of nonnegative independent random variables. Applying Kolmogorov three series theorem we obtain that

$$\mathrm{P}^a \left(\sum_{n=1}^{\infty} Z_n < +\infty \right) = 1 \ ,$$

if

$$\sum_{n=1}^{\infty} \mathrm{P}^a (Z_n > 1) < +\infty \ ,$$

and

$$\sum_{n=1}^{\infty} \mathrm{E}^a \left[Z_n \cdot 1_{\{Z_n \leq 1\}} \right] < +\infty \ .$$

Recall that P^a distribution of Z_n is equal to P^0 distribution of $(1/a_{n+1}^\alpha) \cdot \tau_{a/(n(n+1))}$. It follows that

$$\mathrm{P}^a (Z_n > 1) = \int_{(\frac{a}{n+1})^\alpha}^{+\infty} \frac{\frac{a}{n(n+1)}}{\sqrt{2\pi s^3}} e^{-\frac{\left[\frac{a}{n(n+1)}\right]^2}{2s}} ds \leq$$

$$\leq \frac{a}{\sqrt{2\pi} n(n+1)} \int_{(\frac{a}{n+1})^\alpha}^{+\infty} \frac{ds}{s^{3/2}} = \frac{2a}{\sqrt{2\pi} n(n+1)} \cdot \frac{(n+1)^{\alpha/2}}{a^{\alpha/2}} \ ,$$

and the expectation

$$\mathrm{E}^a \left[Z_n \cdot 1_{\{Z_n \leq 1\}} \right] = \frac{1}{a_{n+1}^\alpha} \int_0^{a_{n+1}^\alpha} s \frac{\frac{a}{n(n+1)}}{\sqrt{2\pi s^3}} e^{-\frac{\left[\frac{a}{n(n+1)}\right]^2}{2s}} ds \leq$$

$$\leq \frac{a(n+1)^{\alpha}}{\sqrt{2\pi}n(n+1)a^{\alpha}} \int_0^{(\frac{a}{n+1})^{\alpha}} \frac{s}{s^{3/2}} ds = \sqrt{\frac{2}{\pi}} \frac{a(n+1)^{\alpha}}{n(n+1)a^{\alpha}} \cdot \frac{a^{\alpha/2}}{(n+1)^{\alpha/2}} \ .$$

Since $\alpha < 2$, then $\alpha/2 = 1 - \varepsilon$, for some $\varepsilon > 0$, and

$$\sum_{n=1}^{\infty} \frac{(n+1)^{1-\varepsilon}}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)^{\varepsilon}} < +\infty \ .$$

Hence, the two series mentioned above are convergent, and we finished the proof of the first part of the theorem.

To prove the second part, it is enough to prove that

$$\int_0^{\tau_0(\omega)} \frac{ds}{B_s^2(\omega)} = +\infty$$

for $\omega \in F$, where $\mathrm{P}^a(F) > 0$. Using the well-known Lévy's modulus of continuity

$$\mathrm{P}^a \left[\limsup_{\varepsilon \to 0} \sup_{0 \leq t_1 < t_2 \leq 1,\ t_2 - t_1 \leq \varepsilon} \frac{| B_{t_2} - B_{t_1} |}{\sqrt{2\varepsilon \ln(1/\varepsilon)}} = 1 \right] = 1 \ ,$$

and the fact that $\mathrm{P}^a(\tau_0 \leq 1) > 0$, we can choose F to be the set of such ω that $\tau_0(\omega) \leq 1$, $B_0(\omega) = a$, and there exists $\delta(\omega) > 0$, such that for every $\varepsilon \leq \delta$

$$\sup_{0 \leq t_1 < t_2 \leq 1,\ t_2 - t_1 \leq \varepsilon} \frac{| B_{t_2}(\omega) - B_{t_1}(\omega) |}{\sqrt{2\varepsilon \ln(1/\varepsilon)}} \leq 2 \ .$$

If $t_2 = \tau_0(\omega)$ and $t_1 = t_2 - \varepsilon$, then it follows, since $B_{t_2}(\omega) = 0$, that, for every $\varepsilon \leq \delta$,

$$\frac{B_{\tau_0(\omega)-\varepsilon}(\omega)}{\sqrt{2\varepsilon \ln(1/\varepsilon)}} \leq 2 \ .$$

Hence, we obtain

$$\int_0^{\tau_0(\omega)} \frac{ds}{B_s^2(\omega)} \geq \int_{\tau_0(\omega)-\delta(\omega)}^{\tau_0(\omega)} \frac{ds}{B_s^2(\omega)} \geq$$

$$\geq \frac{1}{4} \int_0^{\delta(\omega)} \frac{d\varepsilon}{\varepsilon \ln(1/\varepsilon)} = \frac{1}{4} \left[-\ln(\ln(1/\varepsilon)) \right] \Big|_0^{\delta(\omega)} = +\infty \ ,$$

for every $\omega \in F$.

Q.E.D.

With these observations we finished our description of one-dimensional superprocesses. Our idea is to use these results in describing more general cases. First we will describe (Id_t, Ψ)-supersemigroups over the general space $(E, \mathcal{E})$. To do so we have to introduce some special measures on the space of measures on $(E, \mathcal{E})$. This is the content of the following section.

3.3 Measures on Rays

Let $(E, \mathcal{E})$ be a measurable space, and M the space of all finite signed measures equipped with σ-algebra $\mathcal{M}$, which is the smallest σ-algebra such that, for every f bounded $\mathcal{E}$-measurable, the mapping $\mu \mapsto \langle \mu, f \rangle$ is measurable. M^+ is the cone of (positive) finite measures, and $\mathcal{M}^+ = \mathrm{M}^+ \cap \mathcal{M}$. As before, $b\mathcal{E}$ ($bp\mathcal{E}$) denote the set of bounded (bounded and positive) $\mathcal{E}$-measurable real-valued functions.

Consider $t_x(da)$, a positive kernel from $(E, \mathcal{E})$ to $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, i.e., for every $G \in \mathcal{B}(\mathbb{R}_+)$, $x \mapsto t_x(G)$ is $\mathcal{E}$-measurable, and, for every $x \in E$, $t_x(da)$ is a σ-finite (positive) measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$. Assume also that for every compact set $K \subseteq \mathbb{R}_+$, $x \mapsto t_x(K)$ is bounded. Using $t_x(da)$ we define $\vartheta : E \times \mathcal{M}^+ \longrightarrow [0, +\infty]$ by

$$\vartheta(x, N) = t_x\{\ a \in \mathbb{R}_+ \mid a\delta_x \in N\ \}\ , \tag{3.24}$$

for every $x \in E$, $N \in \mathcal{M}^+$, where $\delta_x \in \mathrm{M}^+$ is the point mass at x. Assuming that ϑ is a kernel, we can define, for every $\mu \in \mathrm{M}^+$, a measure Θ_μ on $(\mathrm{M}^+, \mathcal{M}^+)$ by

$$\Theta_\mu(N) = \int_E \vartheta(x, N)\, \mu(dx)\ . \tag{3.25}$$

The following proposition shows that this way we obtain a positive σ-finite measure on $(\mathrm{M}^+, \mathcal{M}^+)$.

Proposition 3.3.1 *ϑ is a positive kernel, and Θ_μ is a positive σ-finite measure on $(\mathrm{M}^+, \mathcal{M}^+)$. For every $F : \mathrm{M}^+ \longrightarrow [-\infty, +\infty]$, which is either positive $\mathcal{M}^+$-measurable or bounded $\mathcal{M}^+$-measurable,*

$$\int_{\mathrm{M}^+} F(\nu)\Theta_\mu(d\nu) = \int_E \int_0^{+\infty} F(a\delta_x)\, t_x(da)\, \mu(dx) \quad . \tag{3.26}$$

Proof. Consider the function $H : \mathbb{R}_+ \times E \longrightarrow \mathrm{M}^+$ given by

$$H(a, x) = a\delta_x \quad .$$

We claim that H is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{E}/\mathcal{M}^+$-measurable. It is enough to prove that, for every $f \in bp\mathcal{E}$, $\langle H, f\rangle(a, x) = a \cdot f(x)$ is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{E}/\mathcal{B}(\mathbb{R}_+)$-measurable, which is obviously true. Fix $N \in \mathcal{M}^+$. Then $\vartheta(x, N) = t_x\{a \in \mathbb{R}_+ \mid (a, x) \in H^{-1}(N)\}$, i.e.,

$$x \mapsto \vartheta(x, N) = \int_{\mathbb{R}_+} 1_{H^{-1}(N)}(a, x)\, t_x(da) \quad ,$$

which is $\mathcal{E}^+$-measurable, since $t_x(da)$ is a kernel and $H^{-1}(N) \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{E}$. The fact that $\vartheta(x, \cdot)$ is a positive σ-finite measure follows directly from 3.24 and properties of $t_x(da)$. It implies immediately that Θ_μ is well-defined positive measure on $(\mathrm{M}^+, \mathcal{M}^+)$. To prove that Θ_μ is σ-finite, let us consider a sequence $N_k = \{\, \nu \in \mathrm{M}^+ \mid \nu(E) \leq k \,\}$, $k \in \mathbb{N}$. Obviously $N_k \in \mathcal{M}^+$, and $(N_k; k \geq 1)$ covers M^+. Notice that

$$\vartheta(x, N_k) = t_x([\,0, k]\,) \quad .$$

Boundedness of $x \mapsto t_x([\,0, k]\,)$ implies that $\Theta_\mu(N_k)$ is finite (recall that μ is a finite measure).

Let $F = 1_N$, $N \in \mathcal{M}^+$. Since ϑ is a kernel we obtain

$$\int_{\mathrm{M}^+} 1_N(\nu)\Theta_\mu(d\nu) = \int_E \vartheta(x, N)\, \mu(dx) =$$

$$= \int_E \int_0^{+\infty} 1_N(a\delta_x)\, t_x(da)\, \mu(dx) \quad .$$

Standard measure-theoretic argument shows that 3.26 is valid for any F, which is either positive $\mathcal{M}^+$-measurable or bounded $\mathcal{M}^+$-measurable.

$Q.E.D.$

Definition 3.3.1 *Measure Θ_μ, defined by 3.25, is called* a measure on rays *with parameter μ, and with respect to a kernel $t_x(da)$.*

Notice that 3.26 applied to $F(\nu) = \exp(-\langle \nu, f \rangle)$, $f \in bp\mathcal{E}$, gives the formula for the Laplace transform in the form

$$\int_{M^+} e^{-\langle \nu, f \rangle} \Theta_\mu(d\nu) = \int_E \int_0^{+\infty} e^{-a\delta_x} t_x(da)\, \mu(dx) \quad . \tag{3.27}$$

We will be interested in some special measures on rays. Let $\lambda(x, N)$ be given by 3.24, where $t_x(da)$ is Lebesgue measure, for every $x \in E$. Then the corresponding measure on rays,

$$\Lambda_\mu(N) = \int_E \lambda(x, N) \mu(dx) \quad , \tag{3.28}$$

is called *the Lebesgue measure on rays.*

Remark 3.3.1 In the case $E = \{1\}$, Λ_μ is a multiple of the standard Lebesgue measure on $\mathbb{R}_+$. However, already for $E = \{1, \ldots, n\}$, $n \geq 2$, Λ_μ is a measure which "lives" only on the coordinate axes. Similarly, if E is a topological space, Λ_μ "lives" only on rays $\{ a\delta_x : a \geq 0 \}$ through δ_x, where $x \in \mathrm{supp}(\mu)$.□

Assuming usual conventions $1/+\infty = 0$, $1/0 = +\infty$, $0 \cdot +\infty = 0$, we obtain the Laplace transform of Λ_μ in the following form,

$$\int_{M^+} e^{-\langle \nu, f \rangle} \Lambda_\mu(d\nu) = \int_E \int_0^{+\infty} e^{-a\delta_x}\, da\, \mu(dx) = \int_E \frac{\mu(dx)}{f(x)} \quad , \tag{3.29}$$

where $f \in bp\mathcal{E}$. Using Λ_μ we define *the exponential measure on rays $exp_\mu(h)$*, where $h \in bp\mathcal{E}$, given by

$$exp_\mu(h)(d\nu) = e^{-\langle \nu, h \rangle} \Lambda_\mu(d\nu) \quad . \tag{3.30}$$

Obviously $exp_\mu(h)$ is absolutely continuous with respect to Λ_μ, and its Laplace transform is given by

$$\int_{M^+} e^{-\langle \nu, f\rangle} exp_\mu(h)(d\nu) = \int_{M^+} e^{-\langle \nu, f\rangle} \cdot e^{-\langle \nu, h\rangle} \Lambda_\mu(d\nu) =$$

$$= \int_E \frac{\mu(dx)}{f(x) + h(x)} \quad . \tag{3.31}$$

3.4 (Id_t, Ψ)-Supersemigroups

Let $(E, \mathcal{E})$ be a standard space and $(M^+, \mathcal{M}^+)$ the measurable space of all finite (positive) measures, defined as in the previous section. Let $\Psi(x, \lambda) : E \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ be a branching mechanism, defined by 2.55. If we consider such a Markov process (ξ_t) on E, that $\xi_t \equiv \xi_0$, then the corresponding semigroup (T_t) is actually (Id_t). As we have seen in section 2.2., there exists a unique (Id_t, Ψ)-supersemigroup, denoted by $Q_t^\Psi(\mu, d\nu)$, and determined by its Laplace transform. We will denote the corresponding nonlinear semigroup by (V_t^Ψ) and (Id_t, Ψ)-superprocess by (X_t^Ψ).

The purpose of this section is to describe $Q_t^\Psi(\mu, d\nu)$. We will use one-dimensional superprocesses and supersemigroups in this description. After that, in the next chapter, we will use (Id_t, Ψ)-supersemigroups as "building blocks" in the description of the general case of superprocesses. Recall the notation from section 2.3., where we denoted the point mass at $x \in E$ by δ_x, and the point mass at $\mu \in M^+$ by Δ_μ, i.e., Δ_μ is an example of a probability measure on $(M^+, \mathcal{M}^+)$.

Notice that for every $x \in E$, $\lambda \mapsto \Psi(x, \lambda)$ is a branching mechanism on the one-point set $\{1\}$. We denote it by Ψ_x. Then the corresponding one-dimensional superprocess, denoted by $(\ell_t^{\Psi_x})$, is the time-changed process of the Lévy process $(L_t^{\Psi_x})$ (see 3.20) stopped at zero. We denote the transition probability of $(\ell_t^{\Psi_x})$ by $q_{t,x}(a, db)$, and corresponding non-linear semigroup (which is actually a function in one-dimensional case) by $v_{t,x}(a)$. Then $v_{t,x} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a function which satisfies

the integral equation

$$v_{t,x}(a) = a + \int_0^t \Psi(x, v_{s,x}(a))ds \quad , \tag{3.32}$$

or, equivalently, the differential equation

$$\begin{cases} \frac{\partial}{\partial t} v_{t,x}(a) = \Psi(x, v_{t,x}(a)) \\ v_{0,x}(a) = a \end{cases} \tag{3.33}$$

The Laplace transform of $q_{t,x}(a, db)$ is given by

$$\int_0^{+\infty} \mathrm{e}^{-cb} q_{t,x}(a, db) = \mathrm{e}^{-av_{t,x}(c)} \quad . \tag{3.34}$$

Consider (V_t^Ψ). Recall that, for every $f \in bp\mathcal{E}$,

$$(V_t^\Psi f)(x) = f(x) + \int_0^t \Psi(x, (V_s^\Psi f)(x))\, ds \quad . \tag{3.35}$$

The uniqueness of the solution of 3.35, and of 3.32, shows that, for every $f \in bp\mathcal{E}$ and $x \in E$,

$$(V_t^\Psi f)(x) = v_{t,x}(f(x)) \quad , \tag{3.36}$$

which implies immediately that

$$\int_{\mathrm{M}^+} \mathrm{e}^{-\langle \nu, f \rangle} Q_t^\Psi(\mu, d\nu) = \exp\left(- \int_E v_{t,x}(f(x))\mu(dx) \right) \quad . \tag{3.37}$$

In the special case when $\mu = m \cdot \delta_z$, $m \in \mathbb{R}_+$, $z \in E$, we obtain

$$\begin{aligned} \int_{\mathrm{M}^+} \mathrm{e}^{-\langle \nu, f \rangle} Q_t^\Psi(m\delta_z, d\nu) &= \mathrm{e}^{-mv_{t,z}(f(z))} = \\ &= \int_0^{+\infty} \mathrm{e}^{-f(z)b} q_{t,z}(m, db) \quad , \end{aligned} \tag{3.38}$$

which proves the following simple result.

Proposition 3.4.1 *If (X_t^Ψ) is a (Id_t, Ψ)-superprocess with $X_0^\Psi \equiv m\delta_z$, then*

$$X_t^\Psi = \ell_t^{\Psi_z} \cdot \delta_z \quad , \tag{3.39}$$

where $\ell_0^{\Psi_z} \equiv m$.

Remark 3.4.1 If we think of superprocesses as the distribution of a population of particles, then 3.39 says that, if the entire population has mass m and is positioned at z, then at the moment t, the entire population is still at z, but its mass changes with the law of one-dimensional branching process which we described in sections 3.1. and 3.2. Notice that (ξ_t) in this case is trivial, i.e., does not change the starting position z. We will see later that this, intuitively very acceptable behaviour (where (ξ_t) affects the position of the population independently from Ψ, which affects only the mass of the population), appears in the cases where (ξ_t) is "simple". However, in general case the behaviour is more complicated and is not clear.□

Formula 3.38 can be used to get $Q_t^{\Psi}(\mu, d\nu)$ for some other μ. The idea is to use the branching property 2.121, and limits of the type 2.125, to calculate $Q_t^{\Psi}(\mu, d\nu)$. We can not obtain explicit formulas by this procedure in the most general case. However, in some special cases we can use measures on rays (see section 3.3) to obtain nice explicit formulas for $Q_t^{\Psi}(\mu, d\nu)$. The procedure is described below.

Assume that, for every $t \geq 0$, there is a positive kernel $\Upsilon_{t,x}(da)$ from $(E, \mathcal{E})$ to $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, such that $x \mapsto \Upsilon_{t,x}(K)$ is bounded for every compact K, and, that there is a function $h_t : E \longrightarrow \mathbb{R}_+$, such that

$$-v_{t,x}(c) = \int_0^{+\infty} e^{-cb} \Upsilon_{t,x}(db) - h_t(x) \quad . \tag{3.40}$$

Using the results from section 3.3 we obtain that, for every $t \geq 0$, there is a measure on rays $\Theta_{\mu,t}$ such that its Laplace transform is

$$\int_{M^+} e^{-\langle \nu, f \rangle} \Theta_{\mu,t}(d\nu) = \int_E \int_0^{+\infty} e^{-bf(x)} \Upsilon_{t,x}(db) \mu(dx) =$$

$$= -\langle \mu, v_{t,x}(f(x)) \rangle + \langle \mu, h_t(x) \rangle \quad . \tag{3.41}$$

Recall now that the Laplace transform of the convolution of two measures on the space $(M^+, \mathcal{M}^+)$ is the product of their Laplace transforms (see section 2.5). Then

3.41 and 3.37 imply that

$$Q_t^{\Psi}(\mu, d\nu) = \mathrm{e}^{-\langle \mu, h_t(x)\rangle} \cdot \sum_{n=1}^{\infty} \frac{\Theta_{\mu,t}^{n*}(d\nu)}{n!} \quad . \tag{3.42}$$

Let us consider now particular branching mechanisms Ψ and show how we can apply results obtained in this section.

Case $\Psi(x,\lambda) = -b(x)\lambda - c(x)\lambda^2;\ b \in b\mathcal{E}, \mathrm{c} \in bp\mathcal{E}, c > 0$

Using 3.36 and 3.6 we obtain, for every $f \in bp\mathcal{E}$,

$$(V_t^{\Psi} f)(x) = \frac{f(x)}{\mathrm{e}^{b(x)t} + f(x)c(x)\frac{\mathrm{e}^{b(x)t}-1}{b(x)}} \quad , \tag{3.43}$$

where the quotient $(\mathrm{e}^{b(x)t} - 1)/b(x)$ is defined by 3.7. This is the example which satisfies 3.42. Let $\Theta_{\mu,t} \sim exp_{\eta}(g)$, where $exp_{\eta}(g)$ is defined by 3.30, and η and g are given by

$$\eta(dx) = \frac{\mathrm{e}^{b(x)t}}{\left[c(x)\frac{\mathrm{e}^{b(x)t}-1}{b(x)}\right]^2}\mu(dx) \quad \text{and} \quad g(x) = \frac{\mathrm{e}^{b(x)t}}{c(x)\frac{\mathrm{e}^{b(x)t}-1}{b(x)}} \quad . \tag{3.44}$$

Let a function h_t be given by

$$h_t(x) = \frac{1}{\mathrm{c}(x)\frac{\mathrm{e}^{b(x)t}-1}{b(x)}} \quad . \tag{3.45}$$

Using 3.31 and 3.41, we obtain that $Q_t^{\Psi}(\mu, d\nu)$ is given by 3.42, since

$$-\langle \mu, h_t(x)\rangle + \int_{\mathrm{M}^+} \mathrm{e}^{-\langle \nu, f\rangle}\Theta_{\mu,t}(d\nu) =$$

$$= -\int_E \frac{\mu(dx)}{c(x)\frac{\mathrm{e}^{b(x)t}-1}{b(x)}} + \int_E \frac{\frac{\mathrm{e}^{b(x)t}}{\left[c(x)\frac{\mathrm{e}^{b(x)t}-1}{b(x)}\right]^2}\mu(dx)}{f(x\cdot) + \frac{\mathrm{e}^{b(x)t}}{c(x)\frac{\mathrm{e}^{b(x)t}-1}{b(x)}}} =$$

$$= -\int_E \left[\frac{1}{c(x)\frac{\mathrm{e}^{b(x)t}-1}{b(x)}} - \frac{\mathrm{e}^{b(x)t}}{\left(c(x)\frac{\mathrm{e}^{b(x)t}-1}{b(x)}\right)\left(f(x)c(x)\frac{\mathrm{e}^{b(x)t}-1}{b(x)} + \mathrm{e}^{b(x)t}\right)}\right]\mu(dx) =$$

$$= -\int_E \frac{f(x)\,\mu(dx)}{\mathrm{e}^{b(x)t} + f(x)c(x)\frac{\mathrm{e}^{b(x)t}-1}{b(x)}} \quad .$$

Simple computation shows that in the language of random measures $\Theta_{\mu,t}$ is the Lévy (canonical) measure of $Q_t^{\Psi}(\mu, d\nu)$ (see [37], chapter 6). Notice that, when $c(x) \equiv 0$, then

$$(V_t^{\Psi} f)(x) = f(x)\mathrm{e}^{-b(x)t} \quad , \tag{3.46}$$

and

$$Q_t^{\Psi}(\mu, d\nu) = \Delta_{\mathrm{e}^{-b(x)t}\mu(dx)}(d\nu) \quad , \tag{3.47}$$

which is a special case of 2.86.

Case $\Psi(x,\lambda) = n(x)(1 - \mathrm{e}^{-a(x)\lambda})$; $n, a \in bp\mathcal{E}, a > 0$

Notice that for every $a > 0$, $n > 0$, real numbers, it is not possible that there exists a measure Υ on $\mathbb{R}_+$ such that

$$\int_0^{+\infty} \mathrm{e}^{-\lambda b}\Upsilon(db) = \frac{1}{a} \ln\left[\mathrm{e}^{nat}(\mathrm{e}^{a\lambda} - 1) + 1\right] - h \quad ,$$

since $\lambda = 0$ would imply $\Upsilon(\mathbb{R}_+) = -h < +\infty$, and $\lambda \to +\infty$ would imply $\Upsilon(\{0\}) = +\infty$. Therefore, in this case we can only use formulas 3.36 and 3.38 to get

$$(V_t^{\Psi} f)(x) = \frac{1}{a(x)} \ln\left[\mathrm{e}^{n(x)a(x)t}(\mathrm{e}^{a(x)f(x)} - 1) + 1\right] \quad , \tag{3.48}$$

and

$$Q_t^{\Psi}(m\delta_z, d\nu) = \sum_{k=0}^{\infty} \binom{-\frac{m}{a(z)}}{k} \mathrm{e}^{-mn(z)t} \left(\mathrm{e}^{-n(z)a(z)t} - 1\right)^k \Delta_{(m+ka(z))\delta_z}(d\nu) \quad . \tag{3.49}$$

Let us also mention the case of $E = \{1, \ldots, n\}$. In that case Ψ is determined by $\Psi_1, \ldots, \Psi_2$, and every $\mu \in \mathrm{M}^+ = \mathbb{R}_+^n$ is $\mu = (\mu_1, \ldots, \mu_n)$. Hence, if we denote $Q_t^{\Psi}(\mu_i \delta_{\{i\}}, d\nu)$ by $Q_t^i(\mu_i, d\nu)$, then

$$Q_t^{\Psi}(\mu, d\nu) = Q_t^1(\mu_1, d\nu) * \ldots * Q_t^n(\mu_n, d\nu) \quad . \tag{3.50}$$

Therefore, the (Id_t, Ψ)-superprocess (X_t^{Ψ}) which starts at $\mu \in \mathbb{R}_+^n$ is given by

$$X_t^{\Psi} = \left(\ell_t^{\Psi_1}, \ldots, \ell_t^{\Psi_n}\right) \quad , \tag{3.51}$$

where $(\ell_t^{\Psi_i})$ are *independent* copies of one-dimensional Ψ_i-superprocesses, such that $(\ell_t^{\Psi_i})$ starts at $\mu_i \in \mathbb{R}_+$. The infinitesimal generator $\mathcal{G}$ of (X_t^Ψ) is given by 2.115, i.e.,

$$(\mathcal{G}f)(x) = \sum_{i=1}^{n} x_i \left(c_i \frac{\partial^2 f}{\partial x_i^2} - b_i \frac{\partial f}{\partial x_i} + \int_0^{+\infty} [f(x + \varepsilon_i u) - f(x)] n_i(du) \right) \quad . \tag{3.52}$$

Notice that one of the characteristics of the process (X_t^Ψ) is that for almost all trajectories ω

$$(X_t^\Psi)_i(\omega) = 0 \implies (X_{t+h}^\Psi)_i(\omega) = 0, \ \forall h \geq 0. \tag{3.53}$$

We will use this observation later.

CHAPTER 4
TRANSITION PROBABILITIES OF SUPERPROCESSES

We will apply results from Chapter 3, to describe supersemigroups of general superprocesses. In the case of "simple" Markov processes, like the uniform motion to the right we obtain explicit formulas. However, in more general cases we are not able to say much. Already in the case of Markov chains we can describe supersemigroups only locally with the application of Cameron-Martin formula.

4.1 Trotter-Kato Formula

In this section we restrict ourselves to the assumptions of section 2.4, i.e., E is a locally compact Hausdorff space with a countable base, $E_\triangle$ the one-point compactification of E, (ξ_t) a Markov process with state space $(E, \mathcal{B}(E))$ ($\triangle$ is a cemetery), whose semigroup (T_t) satisfies the Feller property, and a branching mechanism Ψ that satisfies 2.93 and 2.94.

We have seen, in section 2.4, that under these assumptions the superprocess (X_t) is a Feller process with state space $\mathrm{M}_\triangle^+$, and (X_t) actually "lives" on M^+. We denoted its infinitesimal generator by $\mathcal{G}$, and 2.111 shows how $\mathcal{G}$ is defined on its core $Exp(\mathrm{M}_\triangle^+, \mathcal{D}(\tilde{A}))$ (see 2.109), where $\tilde{A}$ is the infinitesimal generator of (T_t) extended to $\mathrm{C}(E_\triangle)$ (see 2.92).

However, in section 3.4 we described (Id_t, Ψ)-superprocess for every standard space $(E, \mathcal{E})$. In particular, we have the description of (Id_t, Ψ)-superprocess under the assumption of this section. Let us denote (Id_t, Ψ)-superprocess by (X_t^Ψ), and its supersemigroup by $(\tilde{Q}_t^\Psi)$. Of course, (X_t^Ψ) is also a Feller process on $\mathrm{M}_\triangle^+$ and

$Exp(\mathrm{M}_{\triangle}^{+}, \mathcal{D}(0))$ is a core for the infinitesimal generator of (X_t^{Ψ}). Since $\mathcal{D}(0) = \mathrm{C}(E_{\triangle})$, it follows that for every $\tilde{A}$,

$$Exp(\mathrm{M}_{\triangle}^{+}, \mathcal{D}(\tilde{A})) \subseteq Exp(\mathrm{M}_{\triangle}^{+}, \mathcal{D}(0)) = Exp(\mathrm{M}_{\triangle}^{+}, \mathrm{C}(E_{\triangle})) \quad .$$

Let us denote the infinitesimal generator of (X_t^{Ψ}) by $\mathcal{G}^{\Psi}$. Then, by 2.111, for every $f \in \mathrm{C}(E_{\triangle})^{+}$ and $\mu \in \mathrm{M}_{\triangle}^{+}$,

$$\left[\mathcal{G}^{\Psi}(F_f)\right](\mu) = -[F_f(\mu)] \cdot \langle \mu, \tilde{\Psi} f \rangle \quad . \tag{4.1}$$

Consider now the case when $\Psi \equiv 0$, i.e., the case of $(T_t, 0)$-superprocess. Following the notation of the section 2.4 we have that $\tilde{\Psi} \equiv 0$, and $\tilde{V}_t \equiv \tilde{T}_t$. Since $\tilde{T}_t$ is linear and positive we have that $\mu\tilde{T}_t$, defined by

$$\langle \mu\tilde{T}_t, f \rangle = \langle \mu, \tilde{T}_t f \rangle \tag{4.2}$$

is a measure in $\mathrm{M}_{\triangle}^{+}$. Therefore, we have an explicit formula for $(\tilde{T}_t, 0)$-supersemigroup. If we denote $(\tilde{T}_t, 0)$-supersemigroup by $(\tilde{Q}_t^{\xi})$ and $(\tilde{T}_t, 0)$-superprocess by (X_t^{ξ}), we obtain that

$$\tilde{Q}_t^{\xi}(\mu, d\nu) = \Delta_{\mu\tilde{T}_t}(d\nu) \quad , \tag{4.3}$$

and

$$X_t^{\xi} = X_0^{\xi}\tilde{T}_t \quad . \tag{4.4}$$

Let us denote the infinitesimal generator of $(\tilde{Q}_t^{\xi})$ by $\mathcal{G}^{\xi}$. Again, section 2.4 shows that $Exp(\mathrm{M}_{\triangle}^{+}, \mathcal{D}(\tilde{A}))$ is a core for $\mathcal{G}^{\xi}$, and, for every $f \in \mathrm{C}(E_{\triangle})^{+} \cap \mathcal{D}(\tilde{A})$, f strictly positive, and $\mu \in \mathrm{M}_{\triangle}^{+}$,

$$\left[\mathcal{G}^{\xi}(F_f)\right](\mu) = -[F_f(\mu)] \cdot \langle \mu, \tilde{A} f \rangle \quad . \tag{4.5}$$

Using 4.1, 4.5, and 2.111, we obtain that

$$\mathcal{G} = \mathcal{G}^{\Psi} + \mathcal{G}^{\xi} \tag{4.6}$$

on $Exp(\mathrm{M}_{\triangle}^{+}, \mathcal{D}(\tilde{A}))$. Since we have the description of $(\tilde{Q}_t^{\Psi})$ (section 3.4), and the description of $(\tilde{Q}_t^{\xi})$ is simple (see 4.3), the equality 4.6 indicates that we can use the well-known consequence of Trotter-Kato formula (see, for example, [47], Corollary 5.5) to obtain an approximation formula for $(\tilde{Q}_t)$.

Theorem 4.1.1 *For every $F \in \mathrm{C}_0(\mathrm{M}_{\triangle}^{+})$, and, for every $t \geq 0$,*

$$\tilde{Q}_t F = \lim_{n\to\infty} \left(\tilde{Q}_{t/n}^{\Psi}\tilde{Q}_{t/n}^{\xi}\right)^n F = \lim_{n\to\infty} \left(\tilde{Q}_{t/n}^{\xi}\tilde{Q}_{t/n}^{\Psi}\right)^n F \quad , \tag{4.7}$$

and the limit is uniform on bounded t intervals.

Proof. The proof is now almost straightforward application of Corollary 5.5 in [47]. We only have to be careful about the fact that we are dealing with cores for infinitesimal generators, and not with their domains. All three semigroups $(\tilde{Q}_t)$, $(\tilde{Q}_t^{\Psi})$, and $(\tilde{Q}_t^{\xi})$ are C_0 semigroups on Banach space $\mathrm{C}_0(\mathrm{M}_{\triangle}^{+})$, and all three are contractions. Since $\mathcal{D}(\mathcal{G}^{\Psi}) \cap \mathcal{D}(\mathcal{G}^{\xi})$ contains $Exp(\mathrm{M}_{\triangle}^{+}, \mathcal{D}(\tilde{A}))$, it is dense in $\mathrm{C}_0(\mathrm{M}_{\triangle}^{+})$. The image of $Exp(\mathrm{M}_{\triangle}^{+}, \mathcal{D}(\tilde{A}))$ with respect to the operator $zI - \mathcal{G}$, where z is a complex number, such that $Re(z) > 0$, is dense in $\mathrm{C}_0(\mathrm{M}_{\triangle}^{+})$, since $Exp(\mathrm{M}_{\triangle}^{+}, \mathcal{D}(\tilde{A}))$ is the core for $\mathcal{G}$. By 4.6 the same is true with respect to the operator $zI - (\mathcal{G}^{\Psi} + \mathcal{G}^{\xi})$. It follows that the closure

$$\overline{\mathcal{G}^{\Psi} + \mathcal{G}^{\xi}}$$

of $\mathcal{G}^{\Psi} + \mathcal{G}^{\xi}$ is the infinitesimal generator of a contraction C_0 semigroup, which is given by limit in 4.7. By 4.6 and 2.109 we obtain that

$$\mathcal{D}(\mathcal{G}) \subseteq \mathcal{D}(\overline{\mathcal{G}^{\Psi} + \mathcal{G}^{\xi}})$$

and

$$\overline{\mathcal{G}^{\Psi} + \mathcal{G}^{\xi}}/\mathcal{D}(\mathcal{G}) = \mathcal{G} \quad . \tag{4.8}$$

However, both $\mathcal{G}$ and $\overline{\mathcal{G}^{\Psi} + \mathcal{G}^{\xi}}$ are infinitesimal generators for C_0 semigroups of contractions. By 4.8 this is possible only if $\mathcal{G} = \overline{\mathcal{G}^{\Psi} + \mathcal{G}^{\xi}}$.

Q.E.D.

We have explicit formulas for $(\tilde{Q}_t^{\Psi})$ for several cases (see section 3.4), and for $(\tilde{Q}_t^{\xi})$ always. In particular, $(\tilde{Q}_t^{\xi})$ has a very simple description (see 4.3). One may hope to find explicit formulas for $(\tilde{Q}_t)$, by using Theorem above. Unfortunately, it does not seem to be so. The following example illustrates how complex these formulas can be in general. In some special cases we will obtain nice explicit formulas in section 4.3.

Example 4.1.1 Let (ξ_t) be a two-state Markov chain, which stays for exponential time in state 1, then jumps to state 2 and stays there forever. More precisely, its semigrouop is

$$T_t = \begin{bmatrix} e^{-t} & 1 - e^{-t} \\ 0 & 1 \end{bmatrix} , \tag{4.9}$$

and its infinitesimal generator is

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} . \tag{4.10}$$

Of course, in this case $E = E_{\triangle} = \{1, 2\}$, and $C(E_{\triangle}) = \mathbb{R}^2$, and $M_{\triangle}^{+} = \mathbb{R}_{+}^{2}$.

Let the branching mechanism be $\Psi(x, \lambda) = -\lambda^2$, i.e., it has constant coefficients. Then the (T_t, Ψ)-superprocess (X_t) is an $\mathbb{R}_{+}^{2}$-valued stochastic process, whose infinitesimal generator is given by (see 2.115),

$$x_1 \frac{\partial^2}{\partial x_1^2} + x_2 \frac{\partial^2}{\partial x_2^2} - x_1 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} . \tag{4.11}$$

In this case we have explicit formulas for (Q_t^{Ψ}) and (Q_t^{ξ}) (we do not need "tilde" notation here, since $E = E_{\triangle}$). Moreover, we can find also an explicit formula for a nonlinear semigroup (V_t), which is the solution of 2.61. However, even in this (relatively simple) case we were not able to extract explicit formula for (Q_t).

Let us compute the explicit formula for (V_t). For every $f = (f_1, f_2)$ in $C(E_\triangle)^+ = \mathbb{R}_+^2$, $V_t f = ((V_t f)_1, (V_t f)_2)$ is also in $\mathbb{R}_+^2$. We claim that $V_t f$ is given by

$$(V_t f)_1 = \frac{f_2}{f_2 t + 1} + \frac{(f_1 - f_2)e^{-t}}{(f_2 t + 1)^2 \left[\int_0^t \frac{e^{-s} ds}{(f_2 s + 1)^2} (f_1 - f_2) + 1 \right]} \tag{4.12}$$

and

$$(V_t f)_2 = \frac{f_2}{f_2 t + 1} \quad . \tag{4.13}$$

Using results from section 2.1, it is enough to check that $V_t f$ is well-defined, positive, and that it satisfies 2.61. There are no problems with the second coordinate 4.13, since this problem is solved already in section 3.1 (see formula 3.8). Thus, we will concentrate on 4.12. If $f_1 - f_2 \geq 0$, then $(V_t f)_1$ is obviously well-defined and positive. Consider the case when $f_1 - f_2 < 0$. We define a real-valued function $h : \mathbb{R}_+ \longrightarrow \mathbb{R}$ by

$$h(t) = \int_0^t \frac{e^{-s} ds}{(f_2 s + 1)^2} (f_1 - f_2) + 1 \quad .$$

Since $f_1 - f_2 < 0$ we obtain, for every $t \geq 0$,

$$h(t) \geq \int_0^t \frac{ds}{(f_2 s + 1)^2} (f_1 - f_2) + 1 = \frac{1}{f_2} \left[1 - \frac{1}{f_2 t + 1} \right] (f_1 - f_2) + 1 =$$

$$= \frac{t}{f_2 t + 1} (f_1 - f_2) + 1 = \frac{f_1 t + 1}{f_2 t + 1} > 0 \quad .$$

Therefore, $(V_t f)_1$ is well-defined. Let us prove that it is positive. By the proof above we obtain

$$e^t (f_2 t + 1)^2 \cdot h(t) \geq e^t (f_2 t + 1)^2 \frac{f_1 t + 1}{f_2 t + 1} \geq$$

$$\geq (f_2 t + 1)(f_1 t + 1) > 0 \quad ,$$

which implies, since $f_1 - f_2 < 0$,

$$(V_t f)_1 \geq \frac{f_2}{f_2 t + 1} + \frac{f_1 - f_2}{(f_2 t + 1)(f_1 t + 1)} =$$

$$= \frac{f_1 + f_1 f_2 t}{(f_2 t + 1)(f_1 t + 1)} > 0 \quad .$$

It remains to be proved that (V_t) satisfies 2.61. Let us define a real-valued function $H : \mathbb{R}_+ \longrightarrow \mathbb{R}$ by

$$H(t) = \frac{1}{(f_2 t + 1)^2 \cdot h(t)} \quad .$$

By differentiating $H(t)$, and using that $H(0) = 1$, we obtain

$$H(t) = 1 - \int_0^t H(s) \left[e^{-s}(f_1 - f_2)H(s) + 2\frac{f_2}{f_2 s + 1} \right] ds \quad .$$

Multiplying both sides by $e^{-t}(f_1 - f_2)$, and then adding $f_2/(f_2 t + 1)$ to both sides, and using the fact that

$$(T_t f)_1 - \int_0^t \left[\frac{f_2}{f_2 s + 1} \right]^2 ds = \frac{f_2}{f_2 t + 1} + e^{-t}(f_1 - f_2) \quad ,$$

we obtain

$$(V_t f)_1 = (T_t f)_1 - \int_0^t \left[\frac{f_2}{f_2 s + 1} \right]^2 ds -$$

$$- \int_0^t e^{s-t} \left\{ \left[e^{-s}(f_1 - f_2)H(s) \right]^2 + 2\frac{f_2}{f_2 s + 1} e^{-s}(f_1 - f_2)H(s) \right\} ds \quad .$$

By changing the variable of integration, and by writing $1 = (1 - e^{-s}) + e^{-s}$, we get

$$(V_t f)_1 = (T_t f)_1 - \int_0^t \left[T_s (V_{t-s} f)^2 \right]_1 ds \quad ,$$

which is exactly 2.61 for the first coordinate in this case.

As we have just seen it was lengthy, but elementary, to check that 4.12 and 4.13 give the solution of 2.61 for our (T_t) and Ψ. However, it was not so easy to find 4.12. To do so we were trying to solve the corresponding differential equation. Combination of several useful suggestions from [38], and many guesses led us to 4.12.

Using results from section 3.4 and formula 4.3 we obtain

$$Q_t^{\xi}((\mu_1, \mu_2); (d\nu_1, d\nu_2)) = \Delta_{(\mu_1 e^{-t}, \mu_1(1-e^{-t}) + \mu_2)}(d\nu_1, d\nu_2) \; , \qquad (4.14)$$

and

$$Q_t^{\Psi}((\mu_1,\mu_2);(d\nu_1,d\nu_2)) = (d\nu_1,d\nu_2)\cdot \mathrm{e}^{-\frac{\mu_1+\mu_2}{t}}\cdot \tag{4.15}$$

$$\cdot\sum_{n=0}^{\infty}\frac{1}{n!}\left[\sum_{k=0}^{n}\binom{n}{k}\frac{\left(\frac{\mu_1}{t^2}\right)^k}{\Gamma(k)}\nu_1^{k-1}\mathrm{e}^{-\nu_1/t}\cdot\frac{\left(\frac{\mu_2}{t^2}\right)^{n-k}}{\Gamma(n-k)}\nu_2^{(n-k)-1}\mathrm{e}^{-\nu_2/t}\right],$$

where Γ is the well-known gamma-function.

These results imply the following formula for the m-th term in Trotter-Kato approximation formula 4.7

$$\left[Q^{\xi}_{t/m}Q^{\Psi}_{t/m}\right]^m((\mu_1,\mu_2);(d\nu_1,d\nu_2)) = (d\nu_1,d\nu_2)\cdot\frac{\mathrm{e}^{-\frac{\nu_1+\nu_2}{t/m}}}{\nu_1\nu_2}\cdot$$

$$\cdot\mathrm{e}^{-\frac{\mu_1+\mu_2}{t/m}}\sum_{n_1=0}^{\infty}\frac{1}{n_1!(t/m)^{2n_1}}\sum_{k_1=0}^{n_1}\binom{n_1}{k_1}\left[\mu_1\mathrm{e}^{-t/m}\right]^{k_1}\left[\mu_1(1-\mathrm{e}^{-t/m})+\mu_2\right]^{n_1-k_1}\cdot$$

$$\cdot\sum_{n_2=0}^{\infty}\frac{(t/2m)^{n_1+n_2}}{n_2!(t/m)^{2n_2}}\sum_{k_2=0}^{n_2}\binom{n_2}{k_2}\left[\mathrm{e}^{-t/m}\right]^{k_2}\left[1-\mathrm{e}^{-t/m}\right]^{n_2-k_2}\cdot$$

$$\cdot\sum_{l_2=0}^{n_2-k_2}\binom{n_2-k_2}{l_2}\frac{1}{(1-\mathrm{e}^{-t/m})^{n_2-k_2-l_2}}\cdot$$

$$\cdot\frac{\Gamma(k_1+k_2+l_2)\,\Gamma(n_1+n_2-(k_1+k_2+l_2))}{\Gamma(k_1)\,\Gamma(n_1-k_1)\,\Gamma(k_2)\,\Gamma(n_2-k_2)}\cdot\cdot\cdot\cdot\cdot$$

$$\cdot\sum_{n_m=0}^{\infty}\frac{(t/2m)^{n_{m-1}+n_m}}{n_m!(t/m)^{2n_m}}\sum_{k_m=0}^{n_m}\binom{n_m}{k_m}\left[\nu_1\mathrm{e}^{-t/m}\right]^{k_m}\left[\nu_2(1-\mathrm{e}^{-t/m})\right]^{n_m-k_m}\cdot$$

$$\cdot\sum_{l_m=0}^{n_m-k_m}\binom{n_m-k_m}{l_m}\frac{1}{(1-\mathrm{e}^{-t/m})^{n_m-k_m-l_m}}\cdot$$

$$\cdot\frac{\Gamma(k_{m-1}+k_m+l_m)\,\Gamma(n_{m-1}+n_m-(k_{m-1}+k_m+l_m))}{\Gamma(k_m)\,\Gamma(n_m-k_m)}\cdot$$

Despite the fact that the formula above seems to show some sort of "regularity", and that we have explicit formula for (V_t), we were not able to recognize if there is an explicit formula for (Q_t). This question remains open.$\diamondsuit$

4.2 Superprocesses over Markov Chains

Let (ξ_t) be a Markov chain with state space $\{1, 2, \ldots, n\}$. It is determined by its semigroup (T_t), i.e., by the infinitesimal generator $A = (a_{ij})_1^n$. As we have seen already in section 2.4 and 2.5 the superprocesses over (ξ_t) are branching Markov processes on $\mathbb{R}_+^n$, whose infinitesimal generators are given in Theorem 2.4.3. The class of branching Markov processes on $\mathbb{R}_+^n$ has been studied by Watanabe in [57] (the paper actually deals with the case $n = 2$, but only to make notation simpler) and by Rhyzhov and Skorokhod in [51]. Of course, at that time (1969) they did not consider the terminology of superprocesses. However, in their papers they characterized completely the infinitesimal generators of branching Markov processes on $\mathbb{R}_+^n$ (see, for example, [57], Theorem 2, pp.449-450). Comparing these formulas to 2.115 we can see that the class of superprocesses over Markov chains is smaller than the class of branching Markov processes (there are no termination coefficients in 2.115 and drift coefficients are restricted by b and A). Even more general class, of branching processes on $\mathbb{R}_+$ with immigration, has been studied in [40].

Following the notation of the previous section, we obtain from 2.115 that (Id_t, Ψ)-superprocess (X_t^Ψ) has the infinitesimal generator $\mathcal{G}^\Psi$, given by

$$(\mathcal{G}^\Psi F)(\mu) = \sum_{i=1}^{n} \mu_i \left[c_i \frac{\partial^2 F}{\partial \mu_i^2} - b_i \frac{\partial F}{\partial \mu_i} + \int_0^\infty [F(\mu + \varepsilon_i u) - F(\mu)] n_i(du) \right] . \qquad (4.16)$$

In particular, it shows that (X_t^Ψ) is equal to the process (see 3.51 and 3.52)

$$(\ell_t^{\Psi_1}, \ell_t^{\Psi_2}, \ldots, \ell_t^{\Psi_n}) , \qquad (4.17)$$

where $(\ell_t^{\Psi_i})$ are independent one-dimensional superprocesses with respect to $\Psi_i(\lambda) = \Psi(i, \lambda)$; $i = 1, \ldots, n$. Recall (chapter 3) that each $(\ell_t^{\Psi_i})$ is the time-changed process of the Lévy process $(L_t^{\Psi_i})$ stopped at zero (see 3.20). In particular, it shows that, for

every $i_0 \in \{1, \ldots, n\}$ (see 3.53),

$$\mathrm{P}\left[\left((X^{\Psi}_{t+h})_{i_0} > 0, \text{ for some } h > 0\right) \mid \left((X^{\Psi}_{t})_{i_0} = 0\right)\right] = 0 \quad . \tag{4.18}$$

Formulas 4.3 and 4.4 give the total description of the process (X^{ξ}_t). It is a deterministic process, whose infinitesimal generator is given by (see 2.115)

$$(\mathcal{G}^{\xi}F)(\mu) = \sum_{i=1}^{n} \mu_i (A \cdot \nabla F)_i = \langle \mu, A \cdot \nabla F \rangle \quad . \tag{4.19}$$

Despite these nice formulas for "component processes" (X^{Ψ}_t) and (X^{ξ}_t), we can not recover (X_t) from them in general. However, in the case of diffusions we can obtain the local behaviour of (X_t) through (X^{Ψ}_t) and the Cameron-Martin formula. The case of diffusion, that we will be concerned with, corresponds to the branching mechanism

$$\Psi(i, \lambda) = -b_i \lambda - c_i \lambda^2 \quad ,$$

where $b_i \in \mathbb{R}$, and $c_i > 0$, for every $i = 1, \ldots, n$. The infinitesimal generator $\mathcal{G}^{\Psi}$ is given by 4.16, where $n_i = 0$, $i = 1, \ldots, n$. The superprocess (X_t) over (ξ_t) is determined by its infinitesimal generator $\mathcal{G}$, given by (see 2.115)

$$(\mathcal{G}F)(\mu) = \sum_{i=1}^{n} \mu_i \left(c_i \frac{\partial^2 F}{\partial \mu_i^2} - b_i \frac{\partial F}{\partial \mu_i} \right) + \langle \mu, A \cdot \nabla F \rangle \quad . \tag{4.20}$$

Although one may think that it is suitable to apply Cameron-Martin formula globaly, it is not so. Formula 4.12 shows that in general

$$\mathrm{P}\left[((X_{t+h})_{i_0} > 0, \text{ for some } h > 0) \mid ((X_t)_{i_0} = 0)\right] > 0 \quad . \tag{4.21}$$

Therefore, it is not possible that P_X is absolutely continuous with respect to $\mathrm{P}_{X^{\Psi}}$. Hence, we have to localize our problem.

In the rest of this section we will work on the canonical versions of diffusions involved. More precisely, the probability space is $\Omega = \mathrm{C}(\mathbb{R}_+, \mathbb{R}^n)$, the set of all

continuous $\mathbb{R}^n$-valued functions on $\mathbb{R}_+$. Coordinate process is (x_t), where $x_t(\omega) = \omega(t)$, for $\omega \in \Omega$, $t \geq 0$. The filtration $(\mathcal{F}_t^0)$ is defined by $\mathcal{F}_t^0 = \sigma(x_s;\, s \leq t)$, $\mathcal{F}_\infty^0 = \sigma(\cup_{t\geq 0}\mathcal{F}_t^0)$. Hence, to determine the process simply means to determine a measure on Ω which corresponds to the process. If the process starts at $x \in \mathbb{R}^n$, we will denote the corresponding measure by P^x. If the state space of a diffusion is a closed set $H \subseteq \mathbb{R}^n$, then $\mathrm{C}(\mathbb{R}_+, H) \subseteq \mathrm{C}(\mathbb{R}_+, \mathbb{R}^n)$ is $\mathcal{F}_\infty^0$ measurable subset, since

$$\mathrm{C}(\mathbb{R}_+, H) = \{\, \omega \in \Omega :\; x_q \in H,\; \forall q \in \mathbb{Q}_+ \,\} \quad .$$

Therefore, when we say that H is a state space for a diffusion, given by $(\mathrm{P}^x;\; x \in H)$, we mean simply that $\mathrm{P}^x(\mathrm{C}(\mathbb{R}_+, H)) = 1$. In particular we can consider canonical versions of (X_t) and (X_t^Ψ). The family of probability measures $(\Pi^x;\; x \in \mathbb{R}_+^n)$ determines (X_t), and the family $(\Pi_\Psi^x;\; x \in \mathbb{R}_+^n)$ determines (X_t^Ψ).

Let us consider the local behaviour. For every $\varepsilon > 0$, and for every $M \in \mathbb{N}$, $\varepsilon < M$, we define $H = H(\varepsilon, M) \subseteq \mathbb{R}_+^n$ by

$$H = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}_+^n \;:\; \varepsilon \leq x_i \leq M,\; \forall i = 1, \ldots, n \right\} \quad . \tag{4.22}$$

Obviously H is a closed set. We denote its interior and frontier by $int(H)$ and $Fr(H)$, respectively. Let us define a stopping time $\tau = \tau(\varepsilon, M)$ by

$$\tau = \inf\{\, t > 0 :\; x_t \in Fr(H) \,\} \quad . \tag{4.23}$$

We will consider the stopped process $(x_{t\wedge\tau};\; t \geq 0)$ with respect to τ. For every $x \in int(H)$, the stopped process will "behave" like a starting process until it hits the boundary $Fr(H)$. Since for every $x \in int(\mathbb{R}_+^n)$, we can find ε and M such that $x \in int(H)$, the stopped processes of our superprocesses will describe their local behaviour.

Let us "glue" all the points in $\mathbb{R}^n$ to the closest point on the boundary $Fr(H)$, i.e., we define a mapping $h : \mathbb{R} \longrightarrow [\varepsilon, M]$ by

$$h(x) = \begin{cases} \varepsilon\,; & x < \varepsilon \\ x\,; & x \in [\varepsilon, M] \\ M\,; & x > M \end{cases} \tag{4.24}$$

Therefore, for $(x_1, \ldots, x_n) \in H$, $(h(x_1), \ldots, h(x_n)) = (x_1, \ldots, x_n)$.

Consider now a field of matrices α and two fields of vectors β and γ, given by

$$\alpha_{ij}(x_1, \ldots, x_n) = \begin{cases} 0 & ;\ \text{if } i \neq j \\ 2c_i h(x_i) & ;\ \text{if } i = j \end{cases} \tag{4.25}$$

$$\beta_i(x_1, \ldots, x_n) = -b_i h(x_i)\ , \tag{4.26}$$

$$\gamma_i(x_1, \ldots, x_n) = \frac{\sum_{k=1}^n a_{ki} h(x_k)}{\alpha_{ii}(x_1, \ldots, x_n)}\ . \tag{4.27}$$

Notice that $x \mapsto \alpha(x)$, $x \mapsto \beta(x)$, $x \mapsto \gamma(x)$, and $x \mapsto \alpha(x)\gamma(x)$ are all Borel measurable and bounded on $\mathbb{R}^n$. Also, for every $x \in \mathbb{R}^n$, $\alpha(x)$ is a diagonal matrix with strictly positive coefficients. In particular, $\alpha(x)$ is symmetric and non-negative. Since both functions, $f(x) = x$ and $g(x) = \sqrt{x}$, satisfy Lipschitz condition when restricted on $[\varepsilon, M]$, we conclude that our matrix field and our vector fields satisfy Lipschitz condition which is sufficient for the existence and uniqueness of the corresponding stochastic differential equations (see [50], Chapter IX). More precisely, there exists a diffusion with state space $\mathbb{R}^n$ whose infinitesimal generator is given by the second order differential operator

$$\frac{1}{2}\sum_{i=1}^n \alpha_{ii}(\cdot)\frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^n \beta_i(\cdot)\frac{\partial}{\partial x_i}\ . \tag{4.28}$$

Let us denote the corresponding family of measures by $(\mathrm{P}_\Psi^x;\ x \in \mathbb{R}^n)$. The restriction of the coefficients of the operator 4.28 on H gives us exactly the coefficients of the infinitesimal generator $\mathcal{G}^\Psi$. Let us consider only the set of measures $(\mathrm{P}_\Psi^x;\ x \in int(H))$, and, with respect to these measures, the stopped process $(x_{t\wedge\tau})$. By Theorem 10.2.

in [16], it is again a standard Markov process, and by the uniqueness of the solution of the stopped martingale problem (see p.216 in [24]), it is equal to the superprocess (X_t^Ψ) stopped at τ. By the same argument there exists a diffusion determined by an operator

$$\frac{1}{2}\sum_{i=1}^{n}\alpha_{ii}(\cdot)\frac{\partial^2}{\partial x_i^2}+\sum_{i=1}^{n}\left[\beta_i(\cdot)+(\alpha(\cdot)\gamma(\cdot))_i\right]\frac{\partial}{\partial x_i} \quad . \tag{4.29}$$

The corresponding family of measures we denote by $(\mathrm{P}^x;\ x\in\mathbb{R}^n)$. Since the operator 4.29 corresponds to $\mathcal{G}$ on H, we conclude that the stopped process $(x_{t\wedge\tau})$ is the realization of the superprocess (X_t) stopped at τ. Therefore, if we can recover $\mathrm{P}^x|\mathcal{F}^0_{t\wedge\tau}$ from $\mathrm{P}^x_\Psi|\mathcal{F}^0_{t\wedge\tau}$, then we will describe the local behaviour of (X_t) through the local behaviour of (X_t^Ψ). As formula 4.17 shows, there is a nice characterization of (X_t^Ψ), and it is easier to describe (X_t^Ψ) than (X_t). The following theorem shows that we can use Cameron-Martin formula to achieve this goal.

Theorem 4.2.1 *For every $t\geq 0$, and $x\in int(H)$, $\mathrm{P}^x|\mathcal{F}^0_{t\wedge\tau}$ is absolutely continuous with respect to $\mathrm{P}^x_\Psi|\mathcal{F}^0_{t\wedge\tau}$, and*

$$\frac{d\mathrm{P}^x}{d\mathrm{P}^x_\Psi}|_{\mathcal{F}^0_{t\wedge\tau}}=\exp\left\{\int_0^{t\wedge\tau}\langle\gamma(x_s),\,d\bar{x}_s\rangle\ -\ \frac{1}{2}\int_0^{t\wedge\tau}\langle\gamma,\alpha\gamma\rangle(x_s)ds\right\}\quad, \tag{4.30}$$

where $\bar{x}_t=x_t-\int_0^t\beta(x_s)ds$.

Proof. The theorem is a straightforward consequence of Theorem 1.10. on p.344, and Proposition 1.3. on p.302 in [50]. It is enough to check that α, β, and $\langle\gamma,\alpha\gamma\rangle$ are bounded fields. We established that already for α and β. Consider

$$\langle\gamma,\alpha\gamma\rangle(x_1,\ldots,x_n)=\sum_{i=1}^{n}\gamma_i(x_1,\ldots,x_n)\cdot(\alpha\gamma)_i(x_1,\ldots,x_n)=$$

$$=\sum_{i=1}^{n}\frac{\left(\sum_{k=1}^{n}a_{ki}h(x_k)\right)^2}{\alpha_{ii}(x_1,\ldots,x_n)}\quad,$$

which is bounded by $(M^2/\varepsilon)n\cdot\sup_{i,j}|a_{ij}|^2<+\infty$.

Q.E.D.

Let us mention at the end of this section that more details on the behaviour of the superprocess (X_t) near the boundary of $\mathbb{R}^n_+$ one can find in [57], pp.462-465.

4.3 Superprocesses over Deterministic Markov Processes

Let $(E, \mathcal{E})$ be a standard space and $(\mathrm{M}^+, \mathcal{M}^+)$ the measurable space of all finite (positive) measures. In this section we will not use the infinitesimal generator, thus there is no need for "tilde"-notation as in section 4.1. However, we will deal with "component processes" (X_t^Ψ) and (X_t^ξ). Therefore, superindex Ψ indicates (Id_t, Ψ)-superprocess (see section 3.4) and superindex ξ indicates $(T_t, 0)$-superprocess (see formulas 4.2 - 4.4). Recall that $V_t^\xi = T_t$, and detailed description of (V_t^Ψ) is given in section 3.4.

The following theorem will be the basis of our analysis in this section. It treats the case when the solution (V_t) of 2.61 for (T_t, Ψ)-superprocess is particularly nice.

Theorem 4.3.1 *If $V_t f = V_t^\Psi(T_t f)$, for every $t \geq 0$ and $f \in bp\mathcal{E}$, then, for every $t \geq 0$ and $\mu \in \mathrm{M}^+$, the measure $Q_t(\mu, d\nu)$ is the image measure of $Q_t^\Psi(\mu, d\nu)$ with respect to the mapping*

$$\nu \mapsto \nu T_t \quad , \tag{4.31}$$

where $\nu T_t \in \mathrm{M}^+$ is defined by $\langle \nu T_t, f \rangle = \langle \nu, T_t f \rangle$. In particular, we obtain that the (T_t, Ψ)-superprocess (X_t) is given by

$$X_t = X_t^\Psi T_t \quad . \tag{4.32}$$

Proof. Consider the mapping $\mathrm{M}^+ \ni \nu \mapsto \nu T_t \in \mathrm{M}^+$. To prove that this mapping is $\mathcal{M}^+/\mathcal{M}^+$-measurable, it is enough to prove that

$$\nu \mapsto \langle \nu T_t, f \rangle$$

is $\mathcal{M}^+$-measurable, for every $f \in bp\mathcal{E}$. Since $T_t f \in bp\mathcal{E}$, and $\langle \nu T_t, f \rangle = \langle \nu, T_t f \rangle$, our statement is true.

Recall that Laplace transform determines the measure uniquely. Let us denote the image measure of $Q_t^\Psi(\mu, d\nu)$, with respect to $\nu \mapsto \nu T_t$, by $\bar{Q}$. It is enough to prove that the Laplace transform of $\bar{Q}$ is equal to the Laplace transform of $Q_t(\mu, d\nu)$. Using our assumption on (V_t) we obtain

$$\int_{M^+} e^{-\langle \nu, f\rangle} \bar{Q}(d\nu) = \int_{M^+} e^{-\langle \nu T_t, f\rangle} Q_t^\Psi(\mu, d\nu) =$$
$$= \int_{M^+} e^{-\langle \nu, T_t f\rangle} Q_t^\Psi(\mu, d\nu) = e^{-\langle \mu, V_t^\Psi(T_t f)\rangle} =$$
$$= e^{-\langle \mu, V_t f\rangle} = \int_{M^+} e^{-\langle \nu, f\rangle} Q_t(\mu, d\nu) \quad .$$

$Q.E.D.$

If we combine Proposition 3.4.1 with the theorem above, we will obtain the following formula.

Corollary 4.3.1 *If $V_t f = V_t^\Psi(T_t f)$ and $X_0 \equiv m\delta_z$, $m \in \mathbb{R}_+$, $z \in E$, then*

$$X_t = \ell_t^{\Psi_z} \cdot (\delta_z T_t) \quad , \tag{4.33}$$

where $(\ell_t^{\Psi_z})$ is the one-dimensional branching superprocess, which corresponds to the branching mechanism $\lambda \mapsto \Psi(z, \lambda)$, and is such that $\ell_0^{\Psi_z} \equiv m$. Notice also that the measure $\delta_z T_t \in M^+$ satisfies, for every $C \in \mathcal{E}$,

$$(\delta_z T_t)(C) = P^z(\, \xi_t \in C \,) \quad . \tag{4.34}$$

Formula 4.33 is very explicit, but it is obtained under very strong assumptions. We will show now that deterministic Markov processes and branching mechanism with constant coefficients satisfy all these assumptions.

Let $\Psi(x, \lambda)$ be a branching mechanism on $(E, \mathcal{E})$ with constant coefficients, i.e., Ψ is just the function of λ and does not depend on x. Let (ξ_t) be a deterministic Markov process with state space $(E, \mathcal{E})$, i.e., (ξ_t) is described completely by a function

$$\xi : \mathbb{R}_+ \times E \longrightarrow E \quad ,$$

such that $\xi_t \equiv \xi(t,x)$, if $\xi_0 \equiv x$. The Markov property simply means that, for every $t,s \in \mathbb{R}_+$ and $x \in E$,

$$\xi(t+s,x) = \xi(t,\xi(s,x)) \quad . \tag{4.35}$$

In such a case we have, for every $f \in bp\mathcal{E}$,

$$(T_t f)(x) = f(\xi(t,x)) \quad . \tag{4.36}$$

Consider $V_t : bp\mathcal{E} \longrightarrow bp\mathcal{E}$, defined by

$$(V_t f)(x) = \left[V_t^{\Psi}(T_t f)\right](x) = v_t(f(\xi(t,x))) \quad , \tag{4.37}$$

where v_t is defined by 3.32 and 3.33, and does not depend on $x \in E$, since Ψ has constant coefficients. Hence, $v_t : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a function which satisfies

$$v_t(a) = a + \int_0^t \Psi(v_s(a))ds \quad , \tag{4.38}$$

for every $a \in \mathbb{R}_+$. It follows that

$$T_{t-s}\left[\Psi(V_s f)\right](x) = \left[\Psi(V_s f)\right](\xi(t-s,x)) = \text{by } 4.37 =$$

$$= \Psi\left[v_s[f(\xi(s,\xi(t-s,x)))]\right] = \text{by } 4.35 =$$

$$= \Psi\left[v_s[f(\xi(t,x))]\right] \quad .$$

Using 4.38 and 4.36 we obtain

$$\int_0^t T_{t-s}\left[\Psi(V_s f)\right](x)ds = \int_0^t \Psi\left[v_s[f(\xi(t,x))]\right]ds =$$

$$= v_t[f(\xi(t,x))] - f(\xi(t,x)) = (V_t f)(x) - (T_t f)(x) \quad .$$

Therefore, (V_t) satisfies conditions of Theorem 4.3.1. For every $\mu \in \mathrm{M}^+$ we will denote the image measure of μ, with respect to the mapping $x \mapsto \xi(t,x)$, by $\mu[\xi : t]$. It follows that:

Corollary 4.3.2 *The Dawson-Watanabe superprocess* (X_t) *over deterministic Markov process* (ξ_t) *with branching mechanism* $\lambda \mapsto \Psi(\lambda)$ *with constant coefficients is given by*

$$X_t = X_t^{\Psi}[\xi : t] \quad . \tag{4.39}$$

In particular, if $X_0 \equiv m\delta_z$, $m \in \mathbb{R}_+$, $z \in E$, *then*

$$X_t = \ell_t^{\Psi} \cdot \delta_{\xi(t,z)} = \langle X_t, 1 \rangle \cdot \delta_{\xi(t,z)} \quad , \tag{4.40}$$

where $\ell_0^{\Psi} \equiv m$.

Proof. This is a direct consequence of 4.32 and 4.33. We have to prove only that, for any $\mu \in \mathrm{M}^+$, $\mu T_t = \mu[\xi : t]$, and, in case $\mu = \delta_z$, $\mu T_t = \delta_{\xi(t,z)}$. This follows easily by simple computation

$$\langle \mu T_t, f \rangle = \langle \mu, T_t f \rangle = \langle \mu, f(\xi(t, \cdot)) \rangle = \langle \mu[\xi : t], f \rangle$$

and, if $\mu = \delta_z$,

$$\langle \mu T_t, f \rangle = \int_E f(\xi(t,x))\delta_z(dx) = f(\xi(t,z)) = \langle \delta_{\xi(t,z)}, f \rangle \ .$$

Q.E.D.

Remark 4.3.1 Notice that 4.39 and 4.40 give a total description of the superprocesses over deterministic Markov processes (ξ_t) with branching mechanism $\Psi(\lambda)$. Particularly pleasing is the formula 4.40, since it has very nice heuristic interpretation. If we think of superprocesses as the distribution of a population of particles, then 4.40 says that, if the entire population has mass m and is positioned at $z \in E$, then at the moment t the entire population will be positioned at $\xi(t,z) \in E$, and its mass will be ℓ_t^{Ψ}, which is the one-dimensional branching superprocess with respect to Ψ, described completely in sections 3.1 and 3.2. In other words, the "spatial motion" of

(X_t) is governed by (ξ_t), while the "distribution of mass" is governed by Ψ. Compare this remark to Remark 3.4.1.□

Let us consider now the uniform motion to the right on the real line, i.e., $\xi(t,x) = x + t$. Formula 4.40 in this case states

$$X_t = \ell_t^{\Psi} \cdot \delta_{z+t} = \langle X_t, 1\rangle \cdot \delta_{z+t} \quad . \tag{4.41}$$

However, we can not apply this formula in the case of nonconstant coefficients of Ψ, i.e., $\Psi(x,\lambda)$. Still, even in this case we can obtain explicit formulas for (V_t) in some cases of Ψ. We actually obtained these formulas through the application of Trotter-Kato formula. Once we have them, it is elementary to check that they satisfy proper integral equation. Therefore, we will just list these formulas here.

In the case of $\Psi(x,\lambda) = -b(x)\lambda$, where b is bounded, Borel measurable,

$$(V_t f)(x) = f(x+t) \cdot \mathrm{e}^{-\int_0^t b(x+s)ds} \quad . \tag{4.42}$$

In the case of $\Psi(x,\lambda) = -c(x)\lambda^2$, where $c(x) \geq 0$ is bounded, Borel measurable,

$$(V_t f)(x) = \frac{f(x+t)}{1 + f(x+t)\int_0^t c(x+s)ds} \quad . \tag{4.43}$$

In the case of $\Psi(x,\lambda) = n(x)[\,1 - \mathrm{e}^{-a\lambda}]$, where $n(x) \geq 0$ is bounded, Borel measurable and $a > 0$,

$$(V_t f)(x) = \frac{1}{a} \ln \left[\mathrm{e}^{a\int_0^t n(x+s)ds} \left(\mathrm{e}^{af(x+t)} - 1 \right) + 1 \right] \quad . \tag{4.44}$$

Natural question is now, can we expect that in this cases we will also have

$$X_t = \langle X_t, 1\rangle \cdot \delta_{z+t}, \text{ if } X_0 \equiv \delta_z \quad . \tag{4.45}$$

The following computation, with $g(x) \equiv f(z+t) = \text{const.}$,

$$\mathrm{E}^{\delta_z}\left[\mathrm{e}^{-\langle \langle X_t, 1\rangle \cdot \delta_{z+t}, f\rangle} \right] = \mathrm{E}^{\delta_z}\left[\mathrm{e}^{-\langle X_t, 1\rangle \cdot f(z+t)} \right] =$$

$$= \mathrm{E}^{\delta_z}\left[\mathrm{e}^{-\langle X_t, g\rangle}\right] = \mathrm{e}^{-(V_t g)(z)} \quad ,$$

shows that 4.45 is satisfied in all three cases above, since for all of them

$$(V_t g)(z) = (V_t f)\dot{(}z) \quad .$$

However, the process $\langle X_t, 1\rangle$ is $\mathbb{R}_+$-valued stochastic process, but it is not necessarily a branching process any more (as we have seen in Example 3.1.2). The following example shows that it is also not necessarily a Markov process any more.

Example 4.3.1 Let $\Psi(x,\lambda) = -b(x)\lambda$. If $X_0 \equiv m\delta_z$, we can apply the results from section 2.3, on Kac semigroup, to get

$$\langle X_t, f\rangle = mf(z+t)\cdot \mathrm{e}^{-\int_0^t b(z+u)du} \quad . \tag{4.46}$$

It follows that $\langle X_t, 1\rangle$ is a deterministic process, say (H_t^m), where $H_0^m = m$, and

$$H_t^m = m\cdot \mathrm{e}^{-\int_0^t b(z+u)du} \quad .$$

Suppose otherwise that H describes a Markov process. In that case

$$H_{t+s}^m = H_t^{H_s^m} \quad .$$

By 4.46 it would imply

$$\int_0^t b(z+u)du + \int_0^s b(z+u)du = \int_0^{t+s} b(z+u)du \quad ,$$

which is not true for every choice of b.$\diamondsuit$

As the last result of this thesis we will list the supersemigroups (Q_t) for the cases given in 4.42-4.44.

The case 4.42 is the linear case, thus (Q_t) is given by 2.86 in the section on Kac semigroup. To resolve the case 4.43 we follow the analysis in sections 3.3 and 3.4. We define a kernel

$$\lambda^t(x,N) = \lambda\{\, a\in\mathbb{R}_+ |\ a\cdot\delta_{x+t}\in N\,\} \quad ,$$

where λ is the Lebesgue measure. It is completely analogous to 3.24, except it lives on "shifted rays" δ_{x+t}, and not on δ_x as in 3.24. Then, as in 3.28, we define Λ_μ^t with respect to λ^t, and, as in 3.30 $exp_\mu^t(h)$ with respect to Λ_μ^t. We obtain that

$$Q_t(\mu, d\nu) = \mathrm{e}^{-\langle \mu, h_t \rangle} \sum_{n=0}^{\infty} \frac{\Theta_{\mu,t}^{n*}(d\nu)}{n!} \quad , \tag{4.47}$$

where

$$h_t(x) = \frac{1}{\int_0^t c(x+s)ds} \quad , \tag{4.48}$$

and

$$\Theta_{\mu,t} = exp^t_{[h_t(x)]^2 \mu(dx)} \left(\int_0^t c(x-(t-s))ds \right) \quad . \tag{4.49}$$

Finally, using Newton's binomial formula we can easily compute that in the case 4.44, the supersemigroup (Q_t) for the starting measure $\mu = m\delta_z$, $m > 0$, $z \in \mathbb{R}$, is given by

$$Q_t(m\delta_z, d\nu) = \tag{4.50}$$

$$= \sum_{k=0}^{\infty} \binom{-\frac{m}{a}}{k} \mathrm{e}^{-m\int_0^t n(z+s)ds} \left(\mathrm{e}^{-a\int_0^t n(z+s)ds} - 1 \right)^k \Delta_{(m+ka)\delta_{z+t}}(d\nu) \quad .$$

• • •

CHAPTER 5
CONCLUSION

The results of this thesis can be divided roughly into two groups. The first group consists of results given in chapter 2, while the second group consits of results given in chapters 3 and 4.

The results of the first group are related to the problem of the foundation of superprocesses. As we have seen in the introduction, several ideas were offered in the last twenty years. Our approach relates closely to the approach via Markov semigroups given by Watanabe, Fitzsimmons and Dynkin. What is new in our case is that we consider a superprocess (X_t) as M^+-valued Markov process, where M^+ is the set of positive elements of M, where M is the subspace of the space of linear functionals over a Banach space and linear lattice. In this way we consider a superprocess rather as a functional-valued process, than as a measure-valued process. Of course, in particular examples these functionals will be measures, but our framework, we believe, has several advanteges. First, the conditions are very general on M and (T_t) (see section 2.1), and, although they may appear very technical, are easily checked in more "natural" situations, as is illustrated by Examples 2.1.1 and 2.1.2. Actually this generality goes so far that we think section 2.1 is interesting in itself just from an analytical point of view, as a way of preserving positivity of nonlinearly perturbed semigroups. As we indicated at the end of section 2.1 our condition 2.44 may be even close to being a necessary condition for preservation of positivity. However, in the full generality of section 2.1 this remains an open question.

Secondly, this approach enables us to apply the theory of Banach spaces to superprocesses. The theory of Banach spaces was largely studied and offers variety of mathematical tools. As for the third advantage, let us concentrate on Definition 2.1.3 of positive M-Pettis integrable semigroup. The crucial condition here is that (T_t) is M-invariant. Namely, for particular Markov processes (ξ_t) their Markov semigroups are sometimes invariant with respect to some proper subspaces of the space of measures. In such a case we hope to get (X_t) as M^+-valued process, which can be an important restriction of the state space of (X_t). However, except for applying this to Feller semigroups (T_t) in section 2.4, we did not use this reasoning in the thesis. We hope that in our future work it will be substantially used.

Let us also mention one particular result which belongs to this part. This is Lemma 2.1.3. In order to be able to apply condition 2.44 we need to apply one linear perturbation first (actually Corollary 2.1.1). Several authors noticed that fact in their particular setting, and they usually prove Corollary 2.1.1 by integration by parts or some similar method. Our Lemma 2.1.3 is a much more general result than Corollary 2.1.1, and it is proved as a Fubini-type theorem, which, we believe, is the "true nature" of such results. As one would expect, the results in section 2.3 on Kac semigroup come as an easy consequence of Lemma 2.1.3.

Once we establish superprocesses in a rigorous way, the next question is; what are they? What is their behaviour? How does it change, when we change the parameters given? In spite of the fact that superprocesses were extensively studied in last twenty years, it seems that their behaviour is still not so well known. One usual way to characterize them is through the martingale-type description of their infinitesimal generators. In general, the analysis of Markov processes via infinitesimal generators is one of the most succesful approaches in the theory of Markov processes. The achievements of this approach regarding superprocesses are documented in the

literature mentioned in the introduction and throughout the text. We would like to concentrate on some difficulties here. We see at lest two. The first one is that the space of measures is much less understandable to us than, for example, $\mathbb{R}^n$. In particular, if $\mathbb{R}^n$-valued process is characterized by its infinitesimal generator, then we understand its behaviour, since calculus in $\mathbb{R}^n$ is one of the main tools in its description. We believe that it is much harder to deal with analogous situation in $\mathbf{M}^+$. However, at least in the case of Markov chains, when $\mathbf{M}^+ = \mathbb{R}^n_+$, this type of characterization should be complete, and essentially it was done already by Lamperti and Watanabe in the late sixties and early seventies. But in this case, as well as in the general case, we take note of the second difficulty. Obviously, there are two elements which influence the superprocess. One is the branching mechanism Ψ, another is the Markov process (ξ_t), i.e., its semigroup (T_t). What is the influence of each of them? How do they interact? We believe that these questions are still interesting even on the level of Markov chains (ξ_t).

Here we come to the second group of our results, given in Chapters 3 and 4. Of course, we do not claim that we have answered the questions mentioned above, but we believe that in this thesis we have obtained some improvements in that direction. Our idea was to cocentrate on a careful analysis of a transition supersemigroup $(Q_t(\mu, d\nu))$ or, at least, (V_t). Although it is in general a "more primitive" tool than the infinitesimal generator, we hope, because of the difficulties mentioned above, to understand (X_t) sometimes better in this way. We also tried to obtain explicit formulas, whenever possible. Of course, since it involves the solutions of nonlinear integral equations, we can not expect to get these formulas in general cases. Example 4.1.1 should illustrate problems in that direction.

We started by dividing (X_t) into its "component processes", (X_t^Ψ), which is obtained for general Ψ and $\xi_t \equiv \xi_0$, and (X_t^ξ), which is obtained for $\Psi \equiv 0$ and general

(ξ_t). (X_t^ξ) is trivial, since it is a deterministic process given by 4.4. In Chapter 3 we described (X_t^Ψ) rather completely, by using Lamperti's results on one-dimensional branching processes, which are time-changed Lévy processes. We also improved Lamperti's description by adding some explicit formulas for (V_t) and (Q_t). In the most general case, we have a nice description of (X_t^Ψ), when $X_0^\Psi \equiv m\delta_z$ (Proposition 3.4.1), while for more general starting μ, we use branching property (section 2.5) to get (X_t^Ψ). Particularly interesting is a diffusion case, i.e., the case of $\Psi(x,\lambda) = -b(x)\lambda - c(x)\lambda^2$. In this case, through the introduction of the notion of a measure on rays, we obtained nice explicit formulas for $V_t f$ and $Q_t(\mu, d\nu)$, for every μ.

Let us also mention that in section 3.2 we obtained a nice result on Brownian motion, which is interesting regardless of superprocesses. This is Theorem 3.2.1, which complements "from the left" the Engelbert-Schmidt zero-one law (see Remark 3.2.2).

In Chapter 4 we try to implement the second part of the program, i.e., to relate (X_t) to (X_t^Ψ) and (X_t^ξ). Intuitively, the most interesting results are given in section 4.3, where we give the complete description of (X_t) for deterministic (ξ_t) and Ψ with constant coefficients. The results show that the "spatial motion" of (X_t) depends exclusively on (ξ_t), while the "mass" of (X_t) depends exclusively on Ψ. Explicit formulas are obtained in these cases. Also, explicit formulas are given for uniform motion to the right, and Ψ with nonconstant coefficients.

In general we can not expect such nice formulas, as is illustrated in sections 4.1 and 4.2. In section 4.1 we proved that $Q_t(\mu, d\nu)$ can be approximated by Q_t^Ψ and Q_t^ξ. In section 4.2 we treat the case of Markov chains (ξ_t). It is shown that we can recover local behaviour of (X_t) through local behaviour of (X_t^Ψ), by using Cameron-Martin formula. The information on (ξ_t) is hidden in the density which occurs in the Cameron-Martin formula.

Again Ψ influences the "mass" and (ξ_t) the "spatial motion", but not exclusively. All the time certain parts of the "mass" are shifted from position to position according to (ξ_t). However, except intuitively we don't have any results which will describe this "shifting of mass" more precisely. It is an open question, whether there is some (maybe probabilistic) mechanism to describe this structure? Let us also mention that (X_t) behaves in a totally different way than (X_t^Ψ) at the boundary of $\mathbb{R}^n_+$. When (X_t) hits the boundary it leaves it immediately, unless it is zero, while when (X_t^Ψ) hits the boundary, it remains there forever and eventually ends up in zero.

Finally, since we have the description of (X_t^Ψ) in the most general cases, we hope that in the future we will develop our methods succesfully for more complicated cases of Markov processes (ξ_t).

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BIOGRAPHICAL SKETCH

Hrvoje Šikić was born in Bjelovar, Croatia (former Yugoslavia), on Nov.16, 1959. He was raised in Zagreb, Croatia, where he finished primary school and high school, prior to enrolling at the Department of Mathematics, University of Zagreb, in September 1978. He received diploma degree in mathematics in October 1982.

In October 1982, he entered the postdiploma study at the Department of Mathematics, University of Zagreb. He received Magister of Mathematics degree in February 1986. During that period he worked as a teaching assistant.

The period from March 1986 to March 1987 he spent serving in Yugoslav Army. It was an obligatory service for all male citizens in former Yugoslavia.

He enrolled at the Department of Mathematics, University of Florida, in August 1988 as a graduate student in Ph.D. program. Since then, he has been working as a teaching assistant at the Department of Mathematics, University of Florida.

He has been married to Sonja Radas since June 1988.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Murali Rao, Chairman
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Zoran R. Pop-Stojanović
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Joseph Glover
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

James K. Brooks
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Andrew Rosalsky
Professor of Statistics

This dissertation was submitted to the Graduate Faculty of the Department of Mathematics in the College of Liberal Arts and Sciences and to the Graduate School and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

May 1993

Dean, Graduate School

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